Estimating Volatility Using Intradaily Highs and Lows

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Abstract

We investigate volatility estimators built by summing up quadratic functions of log-prices' intradaily increments, highs, and lows, which we show to be consistent for integrated volatility when log-prices are a Brownian semimartingale. With respect to jumps, we single out four jump-robust estimators as well as estimators that account only for the sum of squared positive (negative) jumps. Next, we derive multivariate CLT's for the $\sqrt{n}$-scaled estimation errors of (some of) these estimators in the presence of leverage effects. Finally we use these CLT's to develop highly efficient jump-robust estimators of integrated volatility and a method for measuring the leverage effect.

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1 Introduction

'Arguably, no concept in financial mathematics is as loosely interpreted and as widely discussed as 'volatility'. [...] 'volatility' has many definitions, and is used to denote various measures of changeability.' [1]

'Volatility is a measure of price variability over some period of time. [...] Volatility can be defined and interpreted in five different ways.' [2]

'Yet, the concept of volatility is somewhat elusive, as many ways exist to measure it and hence to model it.' [3]

Although the above quotations make it clear that 'volatility' is a rather vague concept, volatility estimation has been a very prominent topic in financial econometrics for decades, since many financial applications require knowledge or at least some estimate of price volatility, e.g. asset and derivatives pricing, risk management or portfolio selection. The aim of this paper is to contribute to the literature concerned with estimating volatility in a continuous-time setting. Even in this setting and letting aside implied volatility, volatility may refer to different concepts: we could be interested in the quadratic variation[4] of the log-price process (on some day)

$$\mathrm{QV} := p_1^2 - 2\int_0^1 p_{u-}\, dp_u,$$

where p_t denotes the log-price at time t (on that day) and the trading interval for ease of notation is given by $[0, 1]$. QV coincides with

$$\mathrm{IV} := \int_0^1 \sigma_u^2 du,$$

when log-prices follow a Brownian semimartingale, i.e.

$$p_t = p_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u,$$

[1] Shiryaev (1999, p. 345)

[2] Taylor (2005, p. 189)

[3] Engle & Gallo (2006, p. 4)

[4] The log-prices' quadratic variation is also called 'notional volatility' as well as 'ex-post variation', e.g. Andersen et al. (2002), Barndorff-Nielsen et al. (2008a).

where the drift process μ governs the mean, the process σ (also called (spot or instantaneous) volatility!) accounts for the variability of the returns and W denotes a standard Brownian motion. IV is called integrated variance or integrated volatility[5]. In many cases, one rather wants to estimate IV instead of QV, which, denoting the jumps by $\Delta p_t := p_t - p_{t-}$, and the sum of squared jumps by

$$\mathrm{SSJ} := \sum_{t\in[0,1]} (\Delta p_t)^2,$$

can be decomposed into IV and SSJ:

$$\mathrm{QV} = \mathrm{IV} + \mathrm{SSJ}\,.$$

As IV contains information about the contribution of the continuous part of the log-prices to 'volatility', whereas SSJ can tell us something about the variation of returns due to jumps in prices, one often is interested in disentangling QV into IV and SSJ, see e.g. Barndorff-Nielsen & Shephard (2004), Huang & Tauchen (2005), Barndorff-Nielsen & Shephard (2006), Barndorff-Nielsen et al. (2006b), Veraart (2008).

In many contexts, for instance when considering the value at risk of some financial position, it is not only important to know the risk due to jumps, but to have asymmetric measures of both risks of upward and downward jumps. Theoretically, this amounts to decomposing SSJ into

$$\mathrm{SSJ} = \mathrm{SSpJ} + \mathrm{SSnJ},$$

with the sum of squared positive jumps

$$\mathrm{SSpJ} := \sum_{t\in[0,1]} 1_{\Delta p_t > 0}\,(\Delta p_t)^2$$

and the sum of squared negative jumps

$$\mathrm{SSnJ} := \sum_{t\in[0,1]} 1_{\Delta p_t < 0}\,(\Delta p_t)^2,$$

a topic to which the literature has turned only recently, see Barndorff-Nielsen et al. (2008b).

To summarize, we're interested in estimating QV, IV, SSJ, SSpJ and SSnJ. The easiest task is to estimate quadratic variation, because by definition it is

[5] As is common practice in econometrics, we call σ as well as σ^2 volatility.

the limit of realized variance, at least when there is no microstructure noise[6], where realized variance

$$\mathrm{RV} := \sum_{i=1}^{n} (p_{\frac{i}{n}} - p_{\frac{i-1}{n}})^2$$

equals the sum of squared increments over small intervals.
Much work has recently been devoted to the issue of estimating IV in the presence of jumps, which for instance is possible by using bipower or multipower variation, which are consistent estimators for IV in the presence of jumps, see e.g. Barndorff-Nielsen & Shephard (2004), Barndorff-Nielsen & Shephard (2006), Barndorff-Nielsen et al. (2006a).
With these tools at hand, estimation of $\mathrm{SSJ} = \mathrm{QV} - \mathrm{IV}$ is straightforward by subtracting an estimator of IV, e.g. bipower variation, from an estimator of QV, e.g. realized variance.[7]
Until now, there is only one contribution to the issue of estimating SSpJ and SSnJ: the working paper Barndorff-Nielsen et al. (2008b). There it is shown that the realized semivariances

$$\mathrm{RS}^+ := \sum_{i=1}^{n} 1_{p_{\frac{i}{n}} - p_{\frac{i-1}{n}} > 0} (p_{\frac{i}{n}} - p_{\frac{i-1}{n}})^2,$$

$$\mathrm{RS}^- := \sum_{i=1}^{n} 1_{p_{\frac{i}{n}} - p_{\frac{i-1}{n}} < 0} (p_{\frac{i}{n}} - p_{\frac{i-1}{n}})^2$$

converge to $\frac{1}{2}\,\mathrm{IV} + \mathrm{SSpJ}$ and $\frac{1}{2}\,\mathrm{IV} + \mathrm{SSnJ}$, resp., which easily leads the way to consistently estimating SSpJ and SSnJ by subtracting one-half times bi- or multipower variation.

A different strand of research is concerned with improving volatility estimation by considering high-low-based estimators, where volatility now means the variance of daily returns. Starting with the seminal paper Parkinson (1980), numerous authors have considered different estimators in order to make efficient use of the information contained in the widely available daily high and low prices, see e.g. Garman & Klass (1980), Beckers (1983), Ball & Torous (1984), Rogers & Satchell (1991), Wiggins (1991), Kunitomo (1992), Yang & Zhang (2000). As it has been shown theoretically, by simulations

[6]Recently, a lot of work addresses the issue of microstructure noise, e.g. Zhang et al. (2005), Bandi & Russell (2006), Zhang (2006), Barndorff-Nielsen et al. (2008a).

[7]Nevertheless, deriving CLT's or feasible tests is a demanding task, tackled e.g. by Barndorff-Nielsen & Shephard (2006), Barndorff-Nielsen et al. (2006b), Jacod (2008), Veraart (2008).

as well as empirically (e.g., Taylor (1987), Rogers et al. (1994), Wiggins (1991), Rogers & Satchell (1991), Andersen & Bollerslev (1998), Shu & Zhang (2006)), volatility estimation can be significantly improved by these methods. Because the daily range contains information not incorporated in realized volatility[8] (Engle & Gallo (2006)), several authors have recently proposed models that explicitly model the daily range, e.g. Chou (2005a), Brandt & Jones (2006), Brunetti & Lildholdt (2007), Chen et al. (2008).
The use of the range has also proved fruitful for the estimation of continuous-time models: whereas Gallant et al. (1999) and Alizadeh et al. (2002) use daily ranges to estimate (continuous-time) stochastic volatility models, both Martens & van Dijk (2007) and Christensen & Podolskij (2007) estimate QV by the so-called realized range-based variance[9]

$$\mathrm{RRV} = \frac{1}{4\ln 2}\sum_{i=1}^{n} s_{i,n}^2,$$

where

- $(p^*)_{i,n} := \sup\limits_{\frac{i-1}{n}\leqslant t\leqslant \frac{i}{n}} p_t$ denotes the intradaily high on $[\frac{i-1}{n}, \frac{i}{n}]$,
- $(p_*)_{i,n} := \inf\limits_{\frac{i-1}{n}\leqslant t\leqslant \frac{i}{n}} p_t$ denotes the intradaily low on $[\frac{i-1}{n}, \frac{i}{n}]$, and
- $s_{i,n} := (p^*)_{i,n} - (p_*)_{i,n}$ denotes the intradaily range on $[\frac{i-1}{n}, \frac{i}{n}]$.

It has been shown both theoretically and empirically that using the intradaily ranges significantly improves the estimation of integrated volatility when there are no jumps. Alas, this improvement in efficiency comes at the cost of lacking robustness to jumps.

In this paper, we combine the two strands of the literature discussed above. In particular, in section 2 we examine a whole bunch of estimators, namely all estimators built by summing up quadratic functions of intradaily highs $(p^*)_{i,n}$, lows $(p_*)_{i,n}$, and prices $p_{\frac{i}{n}}$.
We analyze the corresponding six-dimensional vector space of estimators, and span it by estimators designed for special purposes:

[8] Although Chen et al. (2006) call it 'counter-intuitive', it is not too surprising that measures designed for two different concepts of volatility convey different information.

[9] RRV is the realized version of Parkinson's estimator.

- first, we consider four estimators for IV, called realized upside volatility (RUV), realized downside volatility (RDV), realized geometric range-based volatility (RGRV) and realized time-reversed geometric range-based volatility (RTrGRV), and show that these estimators are asymptotically robust not only to finite-activity jumps, but even to the addition of processes of finite variation,

- second, we consider estimators for $\mathrm{IV} + \mathrm{SSpJ}$ and $\mathrm{IV} + \mathrm{SSnJ}$, called realized positive jumps included volatility (RpJV) and realized negative jumps included volatility (RnJV), which are resp. asymptotically robust to negative and positive jumps and to continuous processes of finite variation as well.

The remainder of the paper is organized as follows:
In section 2, we introduce the above-mentioned estimators and analyze some of the properties they have in common, namely their asymptotic robustness to continuous processes of finite variation and their consistency for $\mathrm{QV} = \mathrm{IV}$ when log-prices follow a Brownian semimartingale.
In section 3 we investigate individual properties of the estimators: the asymptotic robustness of RUV, RDV, RGRV and RTrGRV with respect to jumps of finite variation, a consistency result for RUV, RDV, RpJV and RnJV when log-prices are given by a continuous semimartingale, and the asymptotic behaviour of RpJV and RnJV in the presence of jumps.
Section 4 is devoted to multivariate CLT's for the $\sqrt{n}$-scaled estimation errors of the estimators. After discussing some applications of the CLT's, for instance measuring the leverage effect or combining the estimators to highly efficient new ones (thereby recovering the 'realized versions' of some old ones), section 5 concludes, while the appendix contains most of the proofs.

2 Sums of Quadratic Functions of Intradaily Highs, Lows, and Returns

In the following, we will be concerned with estimating 'the volatility' of some log-prices p, where we always (at least) assume that the log-price process $(p_t)_{t\in[0,1]}$ is a semimartingale, i.e. it has càdlàg paths and can be written as the sum of a local martingale and a process of finite variation. For ease of notation, we denote the time interval by $[0, 1]$ and call it a 'day', although it does not necessarily have to correspond to a day. By 'volatility', we mean one of the following four concepts:

- quadratic variation $\mathrm{QV} = p_1^2 - 2\int_0^1 p_- \, dp$,
- integrated volatility $\mathrm{IV} = \int_0^1 \sigma_u^2 du$, when we additionally assume that $p_t = p_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u$ is a Brownian semimartingale,
- integrated volatility IV plus the sum $\mathrm{SSpJ} = \sum_{t \in [0,1]} 1_{\Delta p_t > 0} (\Delta p_t)^2$ of the squared positive jumps: $\mathrm{IV} + \mathrm{SSpJ}$.
- integrated volatility IV plus the sum $\mathrm{SSnJ} = \sum_{t \in [0,1]} 1_{\Delta p_t < 0} (\Delta p_t)^2$ of the squared negative jumps: $\mathrm{IV} + \mathrm{SSnJ}$.

In order to estimate 'volatility', we assume knowledge of some intradaily data, specifically we assume that we are given p_0 and, for some integer n and all $i = 1, \ldots, n$,

- intradaily highs $(p^*)_{i,n} = \sup_{t \in [\frac{i-1}{n}, \frac{i}{n}]} p_t$,
- intradaily lows $(p_*)_{i,n} = \inf_{t \in [\frac{i-1}{n}, \frac{i}{n}]} p_t$,
- intradaily log-prices $p_{\frac{i}{n}}$.

In practical application, these data may either stem from ultra-high-frequent data such as order book data, or from aggregating some very high-frequent data, e.g. of sampling frequency 1 sec, to some lower frequency, e.g. 5 min, which is the most often used procedure in practice to avoid undesired effects caused by microstructure noise, although this is typically not the best way of dealing with microstructure noise (e.g., Zhang et al. (2005))). In the following, we therefore won't elaborate on the issue of microstructure noise, but rather assume that we are given the above-described data, may they come from order book data or from aggregating higher-frequent data.
Given intradialy highs and lows, we consider estimators of the form

$$\sum_{i=1}^{n} f\left((p^*)_{i,n} - p_{\frac{i-1}{n}}, (p_*)_{i,n} - p_{\frac{i-1}{n}}, p_{\frac{i}{n}} - p_{\frac{i-1}{n}}\right), \tag{1}$$

where f is some quadratic function of its three arguments. Clearly the resulting vector space is of dimension 6, and our first goal is to find six 'special'

estimators spanning it.
Building on the estimators

$$V_{\max} = 2(p^* - p_0)(p^* - p_1),\ V_{\min} = 2(p_1 - p_*)(p_0 - p_*)$$

recently introduced by Becker et al. (2007b), we start by introducing realized up- and downside volatility[10], which are the realized versions of $V_{\max}$, $V_{\min}$ and correspond to $f_{\text{RUV}}(a,b,c) := 2a(a-c)$ and $f_{\text{RDV}}(a,b,c) := 2b(b-c)$:

$$\text{RUV} := \sum_{i=1}^{n} 2\left((p^*)_{i,n} - p_{\frac{i-1}{n}}\right)\left((p^*)_{i,n} - p_{\frac{i}{n}}\right),$$

$$\text{RDV} := \sum_{i=1}^{n} 2\left((p_*)_{i,n} - p_{\frac{i-1}{n}}\right)\left((p_*)_{i,n} - p_{\frac{i}{n}}\right).$$

Using the geometric range introduced by Gumbel & Keeney (1950), we further define realized geometric range-based volatility

$$\text{RGRV} := \sum_{i=1}^{n} \frac{1}{2\ln 2 - 1}\left((p^*)_{i,n} - p_{\frac{i-1}{n}}\right)\left(p_{\frac{i-1}{n}} - (p_*)_{i,n}\right)$$

and its time-reversed pendant, realized time-reversed geometric range-based volatility

$$\text{RTrGRV} := \sum_{i=1}^{n} \frac{1}{2\ln 2 - 1}\left((p^*)_{i,n} - p_{\frac{i}{n}}\right)\left(p_{\frac{i}{n}} - (p_*)_{i,n}\right),$$

which correspond to $f_{\text{RGRV}} := \frac{1}{2\ln 2-1}a(-b)$ and $f_{\text{RTrGRV}} := \frac{1}{2\ln 2-1}(a-c)(c-b)$. Finally, we introduce realized positive jumps included volatility RpJV and realized negative jumps included volatility RnJV[11]:

$$\text{RpJV} := \sum_{i=1}^{n} \frac{1}{2}\left(\left((p^*)_{i,n} - p_{\frac{i-1}{n}}\right)^2 + \left(p_{\frac{i}{n}} - (p_*)_{i,n}\right)^2\right),$$

$$\text{RnJV} := \sum_{i=1}^{n} \frac{1}{2}\left(\left((p^*)_{i,n} - p_{\frac{i}{n}}\right)^2 + \left(p_{\frac{i-1}{n}} - (p_*)_{i,n}\right)^2\right),$$

which correspond to $f_{\text{RpJV}}(a,b,c) := \frac{a^2+(b-c)^2}{2}$ and $f_{\text{RnJV}}(a,b,c) := \frac{(a-c)^2+b^2}{2}$. We have the following lemma, whose simple proof we omit:

[10] For the naming of RUV and RDV see Chou (2005b), Becker et al. (2007a).

[11] The reason for baptizing the estimators in this way will become clear later.

Lemma 2.1
The six-dimensional vector space given by (1) *is spanned by* RUV, RDV, RGRV, RTrGRV, RpJV *and* RnJV.

The lemma in particular entails that RV and RRV can be written as linear combinations of these six estimators:

$$\mathrm{RV} = -\frac{1}{2}\,\mathrm{RUV} - \frac{1}{2}\,\mathrm{RDV} + \mathrm{RpJV} + \mathrm{RnJV},$$

$$\mathrm{RRV} = \frac{2\ln 2 - 1}{4\ln 2}\,\mathrm{RGRV} + \frac{2\ln 2 - 1}{4\ln 2}\,\mathrm{RTrGRV} + \frac{1}{4\ln 2}\,\mathrm{RpJV} + \frac{1}{4\ln 2}\,\mathrm{RnJV}\,.$$

We now give a result on the consistency of these estimators for IV, when p follows a Brownian semimartingale.

Theorem 1 *If* $p_t = p_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u$ *with a Brownian motion* W*, locally bounded and predictable drift* μ *and càdlàg (spot) volatility* σ*, then* RUV*,* RDV*,* RGRV*,* RTrGRV*,* RpJV *and* RnJV *(and all convex combinations thereof) converge in probability to* $\mathrm{IV} = \int_0^1 \sigma_u^2 du$*.*

Proof: See appendix.

■

The assumptions of the previous theorem seem somewhat restrictive. Nevertheless, much work is cast within this setting, see e.g. Andersen et al. (2002), Christensen & Podolskij (2007) and the references given there. By Theorem 1, we have a six-dimensional cone of consistent estimators for IV. We now show robustness of these estimators with respect to perturbations in form of continuous processes of finite variation. It holds in a much more general setting than Theorem 1.

Theorem 2 *Suppose that* y *is a continuous process of finite variation and* $\mathrm{RpJV}(p) + \mathrm{RnJV}(p)$ *is bounded in* n *almost surely. Then the differences*

$$\mathrm{RUV}(p+y) - \mathrm{RUV}(p),\ \mathrm{RDV}(p+y) - \mathrm{RDV}(p),$$

$$\mathrm{RGRV}(p+y) - \mathrm{RGRV}(p),\ \mathrm{RTrGRV}(p+y) - \mathrm{RTrGRV}(p),$$

$$\mathrm{RpJV}(p+y) - \mathrm{RpJV}(p),\ \mathrm{RnJV}(p+y) - \mathrm{RnJV}(p)$$

converge in probability to 0*.*

Proof: See appendix.

■

Theorem 2 tells us that all the estimators given by (1) are asymptotically robust to continuous processes of finite variation, whenever boundedness of $\mathrm{RpJV}(p)+\mathrm{RnJV}(p)$ holds, which in particular holds in the setting of Theorem 1. However, Thereom 2 only presumes boundedness of $\mathrm{RpJV}(p) + \mathrm{RnJV}(p)$, which allows it to be applied in much more general situations, for instance when prices are allowed to exhibit jumps.

3 Individual Properties of RUV, RDV, RGRV, RTrGRV, RpJV, and RnJV

Further elaborating on robustness, we now show that RUV, RDV, RGRV, RTrGRV are not only asymptotically robust to continuous processes of finite variation, but to any process of finite variation.

Theorem 3 *If y is a process of finite variation and $\mathrm{RpJV}(p) + \mathrm{RnJV}(p)$ is bounded in n almost surely, the differences*

$$\mathrm{RUV}(p + y) - \mathrm{RUV}(p),\ \mathrm{RDV}(p + y) - \mathrm{RDV}(p),$$

$$\mathrm{RGRV}(p + y) - \mathrm{RGRV}(p),\ \mathrm{RTrGRV}(p + y) - \mathrm{RTrGRV}(p)$$

converge in probability to 0.

Proof: Here we give the arguments for robustness to finitely many jumps, relegating the more tedious details to the appendix. Thus suppose that y exhibits only one jump, occuring at some stopping time T. This jump affects only the subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ with $\frac{i-1}{n} < T \leqslant \frac{i}{n}$, i.e. $i = \lceil nT \rceil$. As

$$(p^*)_{i,n} \to \max(p_{T-}, p_T),\ (p_*)_{i,n} \to \min(p_{T-}, p_T),$$

$$p_{\frac{i-1}{n}} \to p_{T-},\ p_{\frac{i}{n}} \to p_T,$$

the differences will vanish asymptotically because of

$$\left((p^*)_{i,n} - p_{\frac{i-1}{n}}\right)\left((p^*)_{i,n} - p_{\frac{i}{n}}\right) \to (\Delta p_T)^+(\Delta p_T)^- = 0,$$

$$\left((p_*)_{i,n} - p_{\frac{i-1}{n}}\right)\left((p_*)_{i,n} - p_{\frac{i}{n}}\right) \to -(\Delta p_T)^-(-(\Delta p_T)^+) = 0,$$

$$\left((p^*)_{i,n} - p_{\frac{i-1}{n}}\right)\left((p_*)_{i,n} - p_{\frac{i-1}{n}}\right) \to (\Delta p_T)^+(-(\Delta p_T)^-) = 0,$$

$$\left((p^*)_{i,n} - p_{\frac{i}{n}}\right)\left((p_*)_{i,n} - p_{\frac{i}{n}}\right) \to (\Delta p_T)^-(-(\Delta p_T)^+) = 0.$$

■

By Theorem 3, we have robustness of RUV, RDV, RGRV, RTrGRV to all processes of finite variation, in particular to finite-activity jump processes as well as inifinite-activity jump processes of finite variation.
We now assume that p is a semimartingale with continuous paths, and show that RUV (RDV), as soon as it converges, is consistent for QV.[12]

Theorem 4 *Assume that p is a continuous semimartingale. Then the following conditions are equivalent:*

- RUV *converges,*
- $\sum_{i=1}^{n}\left((p^*)_{i,n} - p_{\frac{i-1}{n}}\right)^2$ *converges,*
- $\sum_{i=1}^{n}\left((p^*)_{i,n} - p_{\frac{i}{n}}\right)^2$ *converges,*
- *all the above estimators converge to* QV.

For RDV, $\sum_{i=1}^{n}\left((p_*)_{i,n} - p_{\frac{i-1}{n}}\right)^2$, *and* $\sum_{i=1}^{n}\left((p_*)_{i,n} - p_{\frac{i}{n}}\right)^2$, *an analogous assertion holds.*

Proof: See appendix.

■

We end this section by turning our attention to RpJV, RnJV and explaining their naming.

Theorem 5 RpJV *is robust to negative jumps of finite variation and accounts for the squares of positive jumps, while* RnJV *is robust to positive jumps of finite variation and accounts for the squares of negative jumps.*

Proof: As in the proof of Theorem 3, we have

$$\left((p^*)_{i,n} - p_{\frac{i-1}{n}}\right)^2 \to \left((\Delta p_T)^+\right)^2,$$

[12]It may well be doubted whether there is an analogon to Theorem 4 for RGRV or RTrGRV, as, due to their norming constant $\frac{1}{2\ln 2-1}$, these estimators seem to be somewhat 'less natural' than RUV and RDV, or put differently, it looks as if RGRV and RTrGRV exploit the properties of Brownian motion to a greater extent than RUV and RDV do.

$$\left((p_*)_{i,n} - p_{\frac{i}{n}}\right)^2 \to \left(-(\Delta p_T)^+\right)^2,$$
$$\left((p^*)_{i,n} - p_{\frac{i}{n}}\right)^2 \to \left((\Delta p_T)^-\right)^2,$$
$$\left((p_*)_{i,n} - p_{\frac{i-1}{n}}\right)^2 \to \left(-(\Delta p_T)^-\right)^2,$$

and the proof now follows along the same lines as the proof of Theorem 2.

■

Combining Theorems 4 and 5, we find that for sums of Brownian semimartingales and processes of finite variation, RpJV (RnJV) is a consistent estimator for IV + SSpJ (IV + SSnJ), which explains the name 'realized positive (negative) jumps included volatility'.
Combining the results of this section, we can estimate SSpJ (SSnJ) by subtracting any convex combination of the jump-robust estimators RUV, RDV, RGRV, RTrGRV (or any jump-robust consistent estimator of IV) from RpJV (RnJV).

4 Central Limit Theorems for the Estimation Errors

To be able to construct confidence regions or statistical tests for IV based on the newly introduced estimators, we need a central limit theorem for the properly scaled estimation errors. The aim of this section is to derive such a CLT when there are no jumps in the price process.[13] To achieve this, we have to introduce some stronger assumptions on the spot volatility σ_t. In the sequel, we will assume that

$$p_t = p_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u \tag{2}$$

is a Brownian semimartingale, that σ never vanishes and that it is given by

$$\sigma_t = \sigma_0 + \int_0^t \tilde{\mu}_u du + \int_0^t \tilde{\sigma}_u dW_u + \int_0^t \tilde{v}_u dB_u, \tag{3}$$

where $\tilde{\mu}$, $\tilde{\sigma}$, $\tilde{v}$ are càdlàg, $\tilde{\mu}$ is locally bounded as well as predictable, and B is a Brownian motion independent of W.[14]

[13] The more general case with jumps is delayed for further research.

[14] These are exactly the same assumptions as in Christensen & Podolskij (2007).

We start with a CLT for six related, yet infeasible estimators, which are built using not only the driving Brownian motion W's intradaily increments, but also its highs and lows

$$(W^*)_{i,n} := \sup_{t\in[\frac{i-1}{n},\frac{i}{n}]} W_t\,,\ (W_*)_{i,n} := \inf_{t\in[\frac{i-1}{n},\frac{i}{n}]} W_t.$$

The result is stated using stable convergence in law, which is slightly stronger than mere convergence in law[15].

Theorem 6 *Let p be given by* (2), (3) *and define for*

$$\widehat{\mathrm{V}} \in \{\mathrm{RUV}, \mathrm{RDV}, \mathrm{RGRV}, \mathrm{RTrGRV}, \mathrm{RpJV}, \mathrm{RnJV}\}$$

the auxiliary estimators

$$\widetilde{\widehat{\mathrm{V}}} := \sum_{i=1}^{n} \sigma^2_{\frac{i-1}{n}} f_{\widehat{\mathrm{V}}}\left((W^*)_{i,n} - W_{\frac{i-1}{n}}, (W_*)_{i,n} - W_{\frac{i-1}{n}}, W_{\frac{i}{n}} - W_{\frac{i-1}{n}}\right). \qquad (4)$$

Then we have

$$\sqrt{n}\begin{pmatrix} \widetilde{\mathrm{RUV}} & - & \mathrm{IV} \\ \widetilde{\mathrm{RDV}} & - & \mathrm{IV} \\ \widetilde{\mathrm{RGRV}} & - & \mathrm{IV} \\ \widetilde{\mathrm{RTrGRV}} & - & \mathrm{IV} \\ \widetilde{\mathrm{RpJV}} & - & \mathrm{IV} \\ \widetilde{\mathrm{RnJV}} & - & \mathrm{IV} \end{pmatrix} \xrightarrow{d_S} \frac{8}{3\sqrt{2\pi}} u_{\sigma^2} \int_0^1 \sigma_s^2 dW_s + R \int_0^1 \sigma_s^2 dB_s^{(6)}$$

for a six-dimensional Brownian motion $B^{(6)}$ independent of the filtration $\mathcal{F}$ to which p is adapted, where $u_{\sigma^2} = (0,0,0,0,1,-1)'$ and

$$R = \begin{pmatrix} 1 & & & & & \\ -0.33798 & 0.94115 & & & & \\ 0.26153 & 0.37180 & 0.56332 & & & \\ 0.26153 & 0.37180 & -0.17313 & 0.53606 & & \\ 0.16551 & 0.23529 & 0.11902 & 0.32351 & 0.73917 & \\ 0.16551 & 0.23529 & 0.11902 & 0.16351 & 0.50302 & 0.54161 \end{pmatrix}$$

[15]For details on stable convergenc in law, introduced by Rényi (1963), see e.g. Aldous & Eagleson (1978), Jacod (1997), Barndorff-Nielsen et al. (2008a, Appendix A).

with

$$RR' = \Sigma := \begin{pmatrix} 1 & c1 & c2 & c2 & c3 & c3 \\ c1 & 1 & c2 & c2 & c3 & c3 \\ c2 & c2 & c4 & c5 & c6 & c6 \\ c2 & c2 & c5 & c4 & c6 & c6 \\ c3 & c3 & c6 & c6 & c7 & c8 \\ c3 & c3 & c6 & c6 & c8 & c7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -0.33798 & 0.26153 & 0.26153 & 0.16551 & 0.16551 \\ -0.33798 & 1 & 0.26153 & 0.26153 & 0.16551 & 0.16551 \\ 0.26153 & 0.26153 & 0.52397 & 0.10911 & 0.19781 & 0.19781 \\ 0.26153 & 0.26153 & 0.10911 & 0.52397 & 0.19781 & 0.19781 \\ 0.16551 & 0.16551 & 0.19781 & 0.19781 & 0.67003 & 0.49547 \\ 0.16551 & 0.16551 & 0.19781 & 0.19781 & 0.49547 & 0.67003 \end{pmatrix}$$

with constants

$$c_1 = 1 - 8\ln 2 + \frac{7}{2}\zeta(3),\ c_2 = \frac{\zeta(3)-1}{2(\ln 2 - 1)},\ c_3 = \frac{1}{2} - 2\ln 2 + \frac{7}{8}\zeta(3),$$

$$c_4 = \frac{2(1-2(\ln 2)^2)}{2(\ln 2 - 1)^2},\ c_5 = \frac{-\frac{1}{2} + 2\ln 2 - 4(\ln 2 - 1)^2 + \frac{7}{8}\zeta(3)}{2(\ln 2 - 1)^2},$$

$$c_6 = \frac{-\frac{3}{4} + \frac{11}{16}\zeta(3)}{2(\ln 2 - 1)},\ c_7 = \frac{3}{4} + \frac{7}{8}\zeta(3) - \frac{32}{9\pi},\ c_8 = \frac{3}{4} - 2\ln 2 + \frac{32}{9\pi}$$

and Riemann's Zeta function ζ.[16]

Proof: See appendix.

■

As $\widehat{\widetilde{\mathrm{V}}}$ is infeasible in practice, the next theorem considers the $\sqrt{n}$-scaled difference between the actual volatility estimator $\widehat{\mathrm{V}}$ and its auxiliary version $\widehat{\widetilde{\mathrm{V}}}$.

[16] $\zeta(3) \approx 1.202057$

Theorem 7 *Let p be given by (2), (3). Then we have*

$$\sqrt{n}\begin{pmatrix} \mathrm{RUV} - \widetilde{\mathrm{RUV}} \\ \mathrm{RDV} - \widetilde{\mathrm{RDV}} \\ \mathrm{RGRV} - \widetilde{\mathrm{RGRV}} \\ \mathrm{RTrGRV} - \widetilde{\mathrm{RTrGRV}} \\ \mathrm{RpJV} - \widetilde{\mathrm{RpJV}} \\ \mathrm{RnJV} - \widetilde{\mathrm{RnJV}} \end{pmatrix} \xrightarrow{P} \frac{8}{3\sqrt{2\pi}} \begin{pmatrix} \int_0^1 \sigma_s \tilde{\sigma}_s ds \\ -\int_0^1 \sigma_s \tilde{\sigma}_s ds \\ 0 \\ 0 \\ \int_0^1 \sigma_s \left(\mu_s + \frac{\tilde{\sigma}_s}{2}\right) ds \\ -\int_0^1 \sigma_s \left(\mu_s + \frac{\tilde{\sigma}_s}{2}\right) ds \end{pmatrix}$$

Proof: See appendix.

■

Combining Theorems 6, 7, we have the following result:

Theorem 8 *Let p be given by (2), (3). Then we have*

$$\sqrt{n}\begin{pmatrix} \mathrm{RUV} - \mathrm{IV} \\ \mathrm{RDV} - \mathrm{IV} \\ \mathrm{RGRV} - \mathrm{IV} \\ \mathrm{RTrGRV} - \mathrm{IV} \\ \mathrm{RpJV} - \mathrm{IV} \\ \mathrm{RnJV} - \mathrm{IV} \end{pmatrix} \xrightarrow{d_S} C \begin{pmatrix} \int_0^1 \sigma_s \tilde{\sigma}_s ds \\ -\int_0^1 \sigma_s \tilde{\sigma}_s ds \\ 0 \\ 0 \\ \int_0^1 \sigma_s \left(\mu_s + \frac{\tilde{\sigma}_s}{2}\right) ds + \int_0^1 \sigma_s^2 dW_s \\ -\int_0^1 \sigma_s \left(\mu_s + \frac{\tilde{\sigma}_s}{2}\right) ds - \int_0^1 \sigma_s^2 dW_s \end{pmatrix} + R \int_0^1 \sigma_s^2 dB_s^{(6)}$$

with $C = \dfrac{8}{3\sqrt{2\pi}} \approx 1.063846.$

Theorem 8 can in particular be used in two ways:

- we can estimate the leverage effect by using $\frac{\sqrt{n}}{2C}(\mathrm{RUV} - \mathrm{RDV})$ as an asymptotically unbiased estimator for $\int_0^1 \sigma_s \tilde{\sigma}_s ds$. From Theorem 8, it can easily be derived that the asymptotic distribution of

$$\frac{\sqrt{n}}{2C}(\mathrm{RUV} - \mathrm{RDV}) - \int_0^1 \sigma_s \tilde{\sigma}_s ds$$

is mixed normal $\mathrm{MN}(0, c_L\,\mathrm{IQ})$, with $c_L = \frac{9\pi}{128}(16\ln 2 - 7\zeta(3)) \approx 0.5911$ and integrated quarticity $\mathrm{IQ} := \int_0^1 \sigma_s^4 ds$.

- we can look for 'optimal' convex combinations of the estimators designed for special purposes, where 'optimal' loosely speaking means optimal variance, e.g. we could look for the best jump-robust estimator of IV that is not affected by the presence of leverage as given by $\tilde{\sigma}$.

Defining $u_{\mu,\sigma} := (0,0,0,0,1,-1)'$, $u_{\sigma,\tilde{\sigma}} := (1,-1,0,0,\frac{1}{2},\frac{-1}{2})'$, we can rewrite the limit in Theorem 8 as

$$\frac{8}{3\sqrt{2\pi}}\left(u_{\mu,\sigma}\int_0^1 \sigma_s\mu_s ds + u_{\sigma,\tilde{\sigma}}\int_0^1 \sigma_s\tilde{\sigma}_s ds + u_{\sigma^2}\int_0^1 \sigma_s^2 dW_s\right) + R\int_0^1 \sigma_s^2 dB_s^{(6)},$$

from which it is easily seen that the estimation error of a convex combination $w'(\mathrm{RUV}, \mathrm{RDV}, \mathrm{RGRV}, \mathrm{RTrGRV}, \mathrm{RpJV}, \mathrm{RnJV})'$ will converge stably in law to

$$C\left(w_{\mu,\sigma}\int_0^1 \sigma_s\mu_s ds + w_{\sigma,\tilde{\sigma}}\int_0^1 \sigma_s\tilde{\sigma}_s ds + w_{\sigma^2}\int_0^1 \sigma_s^2 dW_s\right) + \sqrt{w'RR'w}\int_0^1 \sigma_s^2 d\widetilde{B}_s,$$

with $w_{\mu,\sigma} := w'u_{\mu,\sigma}$, $w_{\sigma,\tilde{\sigma}} := w'u_{\sigma,\tilde{\sigma}}$, $w_{\sigma^2} := w'u_{\sigma^2}$ and $\widetilde{B}$ a Brownian motion independent of $\mathcal{F}$. In particular, in case of vanishing $w_{\mu,\sigma}$, $w_{\sigma,\tilde{\sigma}}$, and w_{σ^2}, we will have a mixed normal limit $\mathrm{MN}(0, c_w\,\mathrm{IQ})$ with $c_w := w'RR'w$.

- We obtain RRV by

$$w_{\mathrm{RRV}} = \left(0, 0, \frac{2\ln 2 - 1}{4\ln 2}, \frac{2\ln 2 - 1}{4\ln 2}, \frac{1}{4\ln 2}, \frac{1}{4\ln 2}\right)',$$

  with vanishing $w_{\mu,\sigma}$, $w_{\sigma,\tilde{\sigma}}$, w_{σ^2}, and $c_{w_{\mathrm{RRV}}} = 0.4073322$, thereby regaining the CLT of Christensen & Podolskij (2007).

- In order for the limiting distribution to have zero mean, the weights of RpJV and RnJV as well as those of RUV and RDV must be the same. Optimizing c_w under these restrictions leads to

$$w = (0.201098, 0.201098, 0.190103, 0.190103, 0.108799, 0.108799)',$$

  which happens to be the realized version of the Garman & Klass (1980) estimator, c_w then equals 0.2685811. Alas, as RRV, this estimator will lose consistency for IV in the presence of jumps because it involves RpJV and RnJV.

- Averaging RUV and RDV, we have RUDV $:= \frac{\text{RUV}+\text{RDV}}{2}$ with $c_w = 0.3310109$ and the additional advantage of RUDV being a jump-robust estimator. RUDV turns out to be the realized version of the estimator of Rogers & Satchell (1991).

- Combining RUV, RDV, RGRV, and RTrGRV, we are able to find an even more efficient jump-robust estimator of IV. The optimal convex combination is given by

$$w = (0.2209368, 0.2209368, 0.2790632, 0.2790632, 0, 0)'$$

  with $c_w = 0.292233$, which is a lot better than the corresponding constant 2.61, when using bipower variation as a jump-robust estimator of IV[17].

5 Conclusion

In this paper, we have introduced and analyzed the class of volatility estimators that are built by summing up quadratic functions of intradaily returns, highs, and lows. In particular, we have found several new estimators, four of which are robust to jumps of finite variation, while two others enable us to separately estimate the positive and negative jumps' contribution to quadratic variation. After delivering consistency results under varying assumptions, we proved CLT's for the properly scaled estimation errors of these estimators. These theorems then lead the way to the construction of highly efficient estimators of IV that are robust to jumps. As a further by-product, we found a possibility for estimating the leverage effect.
We leave for further research the issues of extending the CLT's by allowing for jumps in the price process, constructing tests for jumps, finding analogous estimators of integrated quarticity IQ as well as providing small-sample corrected versions of the estimators for finite n.

6 Appendix

6.1 Proof of Theorem 1

First of all, we can without loss of generality restrict μ and σ to be bounded (e.g., Barndorff-Nielsen et al. (2006a)).
The proof proceeds in two steps: in the first step, we show $\widetilde{\widehat{\text{V}}} \xrightarrow{P} \text{IV}$ for all $\widehat{\text{V}} \in$

[17] Of course, bipower variation does not need intradaily highs and lows.

$\{\mathrm{RUV}, \mathrm{RDV}, \mathrm{RGRV}, \mathrm{RTrGRV}, \mathrm{RpJV}, \mathrm{RnJV}\}$ and the auxiliary estimators $\widetilde{\widehat{\mathrm{V}}}$ given by (4). The proof will then be completed by showing that the difference between the infeasible estimator $\widetilde{\widehat{\mathrm{V}}}$ and the feasible volatility estimator $\widehat{\mathrm{V}}$ vanishes asymptotically.

To accomplish the first step, we write $\widetilde{\widehat{\mathrm{V}}} = \sum_{i=1}^{n} \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n}$, with

$$\left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} := \sigma^2_{\frac{i-1}{n}} f_{\widehat{\mathrm{V}}} \left((W^*)_{i,n} - W_{\frac{i-1}{n}}, (W_*)_{i,n} - W_{\frac{i-1}{n}}, W_{\frac{i}{n}} - W_{\frac{i-1}{n}} \right). \tag{5}$$

We have, due to the scaling property of Brownian motion and because $f_{\widehat{\mathrm{V}}}$ is quadratic:

$$E\left(\left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} | \mathcal{F}_{\frac{i-1}{n}} \right) = \frac{1}{n} \sigma^2_{\frac{i-1}{n}} E\, f_{\widehat{\mathrm{V}}} \left((W^*)_1, (W_*)_1, W_1 \right),$$

with $(W^*)_1 := \sup_{t \in [0,1]} W_t$ and $(W_*)_1 := \inf_{t \in [0,1]} W_t$. As it is easy to see[18] that $E\, f_{\widehat{\mathrm{V}}} \left((W^*)_1, (W_*)_1, W_1 \right) = 1$ for all $\widehat{\mathrm{V}}$, we find

$$\sum_{i=1}^{n} E\left(\left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} | \mathcal{F}_{\frac{i-1}{n}} \right) \xrightarrow{P} \mathrm{IV}\,. \tag{6}$$

Using essentially the same arguments, we have for the obviously uncorrelated, zero-mean variables

$$\left(\eta_{\widehat{\mathrm{V}}}\right)_{i,n} := \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} - E\left(\left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} | \mathcal{F}_{\frac{i-1}{n}} \right): \tag{7}$$

$$E\left(\left(\eta_{\widehat{\mathrm{V}}}\right)^2_{i,n} | \mathcal{F}_{\frac{i-1}{n}} \right) = \frac{1}{n^2} \sigma^4_{\frac{i-1}{n}} \operatorname{Var} f_{\widehat{\mathrm{V}}} \left((W^*)_1, (W_*)_1, W_1 \right),$$

which implies

$$\sum_{i=1}^{n} \left(\eta_{\widehat{\mathrm{V}}}\right)_{i,n} \xrightarrow{P} 0. \tag{8}$$

Combining (5) - (8), the first step is complete. In the second step, we will now prove that

$$\widehat{\mathrm{V}} - \widetilde{\widehat{\mathrm{V}}} = \sum_{i=1}^{n} \left\{ f_{\widehat{\mathrm{V}}} \left((p^*)_{i,n} - p_{\frac{i-1}{n}}, (p_*)_{i,n} - p_{\frac{i-1}{n}}, p_{\frac{i}{n}} - p_{\frac{i-1}{n}} \right) \right.$$

$$\left. - f_{\widehat{\mathrm{V}}} \left(\sigma_{\frac{i-1}{n}} \left((W^*)_{i,n} - W_{\frac{i-1}{n}} \right), \sigma_{\frac{i-1}{n}} \left((W_*)_{i,n} - W_{\frac{i-1}{n}} \right), \sigma_{\frac{i-1}{n}} \left(W_{\frac{i}{n}} - W_{\frac{i-1}{n}} \right) \right) \right\}$$

[18] By, e.g., using the generating function given in Garman & Klass (1980).

vanishes asymptotically. To simplify notation, we define

$$a_{i,n} := \sigma_{\frac{i-1}{n}} \left((W^*)_{i,n} - W_{\frac{i-1}{n}} \right), \tilde{a}_{i,n} := (p^*)_{i,n} - p_{\frac{i-1}{n}},$$

$$b_{i,n} := \sigma_{\frac{i-1}{n}} \left((W_*)_{i,n} - W_{\frac{i-1}{n}} \right), \tilde{b}_{i,n} := (p_*)_{i,n} - p_{\frac{i-1}{n}},$$

$$c_{i,n} := \sigma_{\frac{i-1}{n}} \left(W_{\frac{i}{n}} - W_{\frac{i-1}{n}} \right), \tilde{c}_{i,n} := p_{\frac{i}{n}} - p_{\frac{i-1}{n}}.$$

It is then straightforward to show that $|a_{i,n} - \tilde{a}_{i,n}|, |b_{i,n} - \tilde{b}_{i,n}|, |c_{i,n} - \tilde{c}_{i,n}|$ are bounded by

$$\kappa_{i,n} := \sup_{s,t \in [\frac{i-1}{n}, \frac{i}{n}]} \left(\int_s^t \mu_\tau d\tau + \int_s^t \left(\sigma_\tau - \sigma_{\frac{i-1}{n}} \right) dW_\tau \right).$$

Because $f_{\widehat{\mathrm{V}}}$ is quadratic, the difference

$$f_{\widehat{\mathrm{V}}}(\tilde{a}_{i,n}, \tilde{b}_{i,n}, \tilde{c}_{i,n}) - f_{\widehat{\mathrm{V}}}(a_{i,n}, b_{i,n}, c_{i,n})$$

can up to some constants be written as sums of terms of the form

$$d_{i,n}(\tilde{e}_{i,n} - e_{i,n}),\ (\tilde{d}_{i,n} - d_{i,n})(\tilde{e}_{i,n} - e_{i,n}),$$

with $d_{i,n}$, $e_{i,n}$ either $a_{i,n}$, $b_{i,n}$ or $c_{i,n}$ and $\tilde{d}_{i,n}, \tilde{e}_{i,n}$ either $\tilde{a}_{i,n}$, $\tilde{b}_{i,n}$ or $\tilde{c}_{i,n}$. Using the range $s_{i,n} := (W^*)_{i,n} - (W_*)_{i,n}$, it is clear that $a_{i,n}$, $b_{i,n}$ and $c_{i,n}$ are bounded by $\sigma_{\frac{i-1}{n}} s_{i,n}$, while differences $\tilde{d}_{i,n} - d_{i,n}$, $\tilde{e}_{i,n} - e_{i,n}$ are bounded by $\kappa_{i,n}$. In order to complete the proof, we therefore only have to show that the sums

$$\sum_{i=1}^n \sigma_{\frac{i-1}{n}} s_{i,n} \kappa_{i,n}\,, \sum_{i=1}^n \kappa_{i,n}^2$$

converge to 0 in probability, which can be easily seen by applying the Cauchy-Schwarz inequality to the first sum and then using exactly the same arguments as in Christensen & Podolskij (2007, p. 342f).

6.2 Proof of Theorem 2

First of all, because y is of finite variation, it can be written as the difference of two monotonuos continuous processes. We only give the proof for robustness to non-decreasing continuous processes y, as the proof for non-increasing continuous processes can be done in an exactly analogous way. By straightforward arguments,

$$\left| ((p+y)^*)_{i,n} - (p+y)_{\frac{i-1}{n}} - \left((p^*)_{i,n} - p_{\frac{i-1}{n}} \right) \right|,$$

$$|((p+y)_*)_{i,n} - (p+y)_{\frac{i-1}{n}} - \Big((p_*)_{i,n} - p_{\frac{i-1}{n}}\Big)|,$$

$$\Big|(p+y)_{\frac{i}{n}} - (p+y)_{\frac{i-1}{n}} - \Big(p_{\frac{i}{n}} - p_{\frac{i-1}{n}}\Big)\Big|$$

are all bounded by $|y_{\frac{i}{n}} - y_{\frac{i-1}{n}}|$. Using the notations

$$\tilde{a}_{i,n} := ((p+y)^*)_{i,n} - (p+y)_{\frac{i-1}{n}}, a_{i,n} := (p^*)_{i,n} - p_{\frac{i-1}{n}},$$

$$\tilde{b}_{i,n} := ((p+y)_*)_{i,n} - (p+y)_{\frac{i-1}{n}}, b_{i,n} := (p_*)_{i,n} - p_{\frac{i-1}{n}},$$

$$\tilde{c}_{i,n} := (p+y)_{\frac{i}{n}} - (p+y)_{\frac{i-1}{n}}, c_{i,n} := p_{\frac{i}{n}} - p_{\frac{i-1}{n}},$$

the differences $\mathrm{RUV}(p+y) - \mathrm{RUV}(p)$ etc. can be written as

$$\sum_{i=1}^{n}\Big(f(\tilde{a}_{i,n}, \tilde{b}_{i,n}, \tilde{c}_{i,n}) - f(a_{i,n}, b_{i,n}, c_{i,n})\Big).$$

As the functions f are quadratic, using the inequalities established above, up to some constants we end up with sums of the form

$$\sum_{i=1}^{n}|d_{i,n}||y_{\frac{i}{n}} - y_{\frac{i-1}{n}}|, \sum_{i=1}^{n}(y_{\frac{i}{n}} - y_{\frac{i-1}{n}})^2,$$

with $d_{i,n}$ either $a_{i,n}$, $b_{i,n}$, $c_{i,n}$, $a_{i,n} - c_{i,n}$, or $b_{i,n} - c_{i,n}$. Using Cauchy-Schwarz inequality as well as the boundedness of $\mathrm{RpJV}(p) + \mathrm{RnJV}(p)$, the result now follows from the fact that

$$\sum_{i=1}^{n}(y_{\frac{i}{n}} - y_{\frac{i-1}{n}})^2 \xrightarrow{P} 0,$$

because y is continuous and of finite variation.

6.3 Proof of Theorem 3

Due to Theorem 2, we have robustness with respect to continuous processes of finite variation, whereas the part of the proof given in the main text gives us robustness to finitely many jumps. So we're left with proving robustness to pure-jump processes of finite variation, possibly having infinitely many jumps. As in the proof of Theorem 2, we may and will restrict the proof to non-decreasing processes y. Now, for every (fixed) $\varepsilon > 0$, we can decompose y into

- its 'large jumps part' $y^{(l,\varepsilon)}$, with $y_1 - y_1^{(l,\varepsilon)} \leqslant \varepsilon$ and only finitely many jumps, and

- the 'small jumps part' $y^{(s,\varepsilon)} = y - y^{(l,\varepsilon)}$, which may have infinitely many jumps whose sum is bounded by ε.

As we have robustness to finitely many jumps, we only have to consider the 'small jumps part' $y^{(s,\varepsilon)}$. Obviously, we have

$$\left(\left(\left(p + y^{(s,\varepsilon)}\right)^*\right)_{i,n} - \left(p + y^{(s,\varepsilon)}\right)_{\frac{i-1}{n}}\right) - \left((p^*)_{i,n} - p_{\frac{i-1}{n}}\right) < y^{(s,\varepsilon)}_{\frac{i}{n}} - y^{(s,\varepsilon)}_{\frac{i-1}{n}} < \varepsilon,$$

and analoguous formulæ for the infimum and final value in $[\frac{i-1}{n}, \frac{i}{n}]$, which entail, as in the proof of the previous theorem, that we're left, up to some constants, with sums that contain the factor $y^{(s,\varepsilon)}_{\frac{i}{n}} - y^{(s,\varepsilon)}_{\frac{i-1}{n}}$. Using Cauchy-Schwarz inequality, we get (uniformly in n) $\mathrm{RUV}(p + y^{(s,\varepsilon)}) - \mathrm{RUV}(p) \leqslant C\varepsilon$ etc., which completes the proof.

6.4 Proof of Theorem 4

We first prove the following lemma, which itself is of some interest.

Lemma 6.1
For any continuous semimartingale $(X_t)_{t \geqslant 0}$ with $X_0 = 0$, running maximum $(X^*)_t = \sup\limits_{0 \leqslant s \leqslant t} X_s$, and running minimum $(X_*)_t = \inf\limits_{0 \leqslant s \leqslant t} X_s$, we have:

$$2X^*(X^* - X) = (X^*)^2 - 2\int X^* dX,$$

$$2X_*(X_* - X) = (X_*)^2 - 2\int X_* dX.$$

Proof: We only have to prove the first assertion, as the second one follows by applying the first one to $-X$. As X^* is non-decreasing, it is of finite variation, hence $(X^*)^2 = 2\int X^* dX^*$. Because for continuous processes, $X^* - X$ vanishes on the support of dX^*, we further have $\int (X^* - X) dX^* = 0$, which implies $(X^*)^2 = 2\int X dX^*$. By integration by parts, we find due to $[X, X^*] = 0$: $\int X dX^* = X^* X - \int X^* dX$, which entails $(X^*)^2 = 2(X^* X - \int X^* dX)$, from which the assertion follows easily.

■

We now proceed to the proof of Theorem 4. Again we only have to prove the assertion for RUV, as the proof for RDV follows in a completely analogous way. Applying Lemma 6.1 to $X_t := p_{t+\frac{i-1}{n}} - p_{\frac{i-1}{n}}$ for every $i = 1, \dots, n$, we have

$$\mathrm{RUV} = \sum_{i=1}^{n} \left(\left((p^*)_{i,n} - p_{\frac{i-1}{n}} \right)^2 - 2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\left(p^*_{i,n}\right)_t - p_{\frac{i-1}{n}} \right) d(p_t - p_{\frac{i-1}{n}}) \right),$$

where $\left(p^*_{i,n}\right)_t := \sup_{\frac{i-1}{n} \leqslant s \leqslant t} p_s$ denotes the running maximum of p on $[\frac{i-1}{n}, \frac{i}{n}]$. This gives

$$\mathrm{RUV} = \sum_{i=1}^{n} \left((p^*)_{i,n} - p_{\frac{i-1}{n}} \right)^2 - 2 \int_0^1 p^*_n dp,$$

with $(p^*_n)_t := \left(p^*_{1+\lfloor nt \rfloor, n}\right)_t - p_{\frac{\lfloor nt \rfloor}{n}}$. As the integrand converges to 0, the integral $\int_0^1 p^*_n dp$ vanishes asymptotically, showing the equivalence of the convergence of RUV and $\sum_{i=1}^{n} \left((p^*)_{i,n} - p_{\frac{i-1}{n}} \right)^2$ and their convergence to the same limit in case of convergence. Applying this to $\tilde{p}_t := p_{1-t}$, we get the same result for RUV and $\sum_{i=1}^{n} \left((p^*)_{i,n} - p_{\frac{i}{n}} \right)^2$. The result now follows from the decomposition

$$\mathrm{RV} = -\,\mathrm{RUV} + \sum_{i=1}^{n} \left((p^*)_{i,n} - p_{\frac{i-1}{n}} \right)^2 + \sum_{i=1}^{n} \left((p^*)_{i,n} - p_{\frac{i}{n}} \right)^2.$$

6.5 Proof of Theorem 6

First of all, we can, as in the proof of Theorem 1, without loss of generality restrict μ and σ to be bounded. From Barndorff-Nielsen et al. (2006a, (2.17)) we have

$$\sqrt{n} \left(\sum_{i=1}^{n} \frac{1}{n} \sigma^2_{\frac{i-1}{n}} - \mathrm{IV} \right) \xrightarrow{P} 0,$$

which, due to the properties of stable convergence in law, allows us to replace IV by $\sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}}$, i.e. we have to show the stable convergence in law

$$\sqrt{n}\begin{pmatrix} \widetilde{\mathrm{RUV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RDV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RGRV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RTrGRV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RpJV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RnJV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \end{pmatrix} \xrightarrow{d_S} \frac{8}{3\sqrt{2\pi}}\, u_{\sigma^2} \int\limits_0^1 \sigma_s^2 dW_s + R\int\limits_0^1 \sigma_s^2 dB_s^{(6)}.$$

Using for $\widehat{\mathrm{V}} \in \{\mathrm{RUV}, \mathrm{RDV}, \mathrm{RGRV}, \mathrm{RTrGRV}, \mathrm{RpJV}, \mathrm{RnJV}\}$ the $\mathcal{F}_{\frac{i-1}{n}}$-measurable r.v.'s $\left(\eta_{\widehat{\mathrm{V}}}\right)_{i,n}$ given by (7) and defining

$$\begin{aligned} \left(\chi_{\widehat{\mathrm{V}}}\right)_{i,n} &:= \sqrt{n}\left(\eta_{\widehat{\mathrm{V}}}\right)_{i,n} \\ &= \sigma^2_{\frac{i-1}{n}}\sqrt{n}\left(f_{\widehat{\mathrm{V}}}\left((W^*)_{i,n} - W_{\frac{i-1}{n}}, (W_*)_{i,n} - W_{\frac{i-1}{n}}, W_{\frac{i}{n}} - W_{\frac{i-1}{n}}\right) - \frac{1}{n}\right), \end{aligned}$$

we have

$$\sqrt{n}\begin{pmatrix} \widetilde{\mathrm{RUV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RDV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RGRV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RTrGRV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RpJV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \\ \widetilde{\mathrm{RnJV}} & - & \sum\limits_{i=1}^{n} \frac{1}{n}\sigma^2_{\frac{i-1}{n}} \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} \left(\chi_{\mathrm{RUV}}\right)_{i,n} \\ \left(\chi_{\mathrm{RDV}}\right)_{i,n} \\ \left(\chi_{\mathrm{RGRV}}\right)_{i,n} \\ \left(\chi_{\mathrm{RTrGRV}}\right)_{i,n} \\ \left(\chi_{\mathrm{RpJV}}\right)_{i,n} \\ \left(\chi_{\mathrm{RnJV}}\right)_{i,n} \end{pmatrix} =: \sum_{i=1}^{n} \chi_{i,n}.$$

The proof now follows by applying Theorem 3-2 of Jacod (1997), for which we have to show:

$$\sup_t \Big|\sum_{i=1}^{\lfloor nt \rfloor} E(\chi_{i,n}|\mathcal{F}_{\frac{i-1}{n}})\Big| \xrightarrow{P} 0, \tag{9}$$

$$\sum_{i=1}^{\lfloor nt\rfloor}\left(E(\chi_{i,n}\chi_{i,n}'|\mathcal{F}_{\frac{i-1}{n}})-E(\chi_{i,n}|\mathcal{F}_{\frac{i-1}{n}})E(\chi_{i,n}'|\mathcal{F}_{\frac{i-1}{n}})\right)\xrightarrow{P}F_t$$

$$\text{for } F_t:=\left(\frac{32}{9\pi}u_{\sigma^2}u_{\sigma^2}'+RR'\right)\int_0^t\sigma_s^4ds, \quad (10)$$

$$\sum_{i=1}^{\lfloor nt\rfloor}E\left(\chi_{i,n}\left(W_{\frac{i}{n}}-W_{\frac{i-1}{n}}\right)|\mathcal{F}_{\frac{i-1}{n}}\right)\xrightarrow{P}\frac{8}{3\sqrt{2\pi}}u_{\sigma^2}\int_0^t\sigma_s^2ds, \quad (11)$$

$$\sum_{i=1}^{\lfloor nt\rfloor}E\left(|\chi_{i,n}|^21_{\{|\chi_{i,n}|>\varepsilon\}}|\mathcal{F}_{\frac{i-1}{n}}\right)\xrightarrow{P}0\ \forall\varepsilon>0, \quad (12)$$

$$\sum_{i=1}^{\lfloor nt\rfloor}E\left(\chi_{i,n}\left(N_{\frac{i}{n}}-N_{\frac{i-1}{n}}\right)|\mathcal{F}_{\frac{i-1}{n}}\right)\xrightarrow{P}0\ \forall N\in\mathcal{M}_b \text{ with } [W,N]=0. \quad (13)$$

1. (9) holds due to $E\left(\chi_{i,n}|\mathcal{F}_{\frac{i-1}{n}}\right)=\sqrt{n}E\left(\eta_{i,n}|\mathcal{F}_{\frac{i-1}{n}}\right)=0$, as seen in the proof of Theorem 1.

2. To prove (10), we only have to consider $\sum_{i=1}^{\lfloor nt\rfloor}E\left(\chi_{i,n}\chi_{i,n}'|\mathcal{F}_{\frac{i-1}{n}}\right)$, because $E\left(\chi_{i,n}|\mathcal{F}_{\frac{i-1}{n}}\right)=0$. As $f_{\widehat{\mathrm{V}}}$ is quadratic, we find, by using the scaling property of Brownian motion, that the components of this matrix are given by

$$n\sum_{i=1}^{\lfloor nt\rfloor}E\left(\left(\eta_{\widehat{\mathrm{V}}_1}\right)_{i,n}\left(\eta_{\widehat{\mathrm{V}}_2}\right)_{i,n}|\mathcal{F}_{\frac{i-1}{n}}\right)=E\left(\widetilde{\eta}_{\widehat{\mathrm{V}}_1}\widetilde{\eta}_{\widehat{\mathrm{V}}_2}\right)\sum_{i=1}^{\lfloor nt\rfloor}\frac{1}{n}\sigma^4_{\frac{i-1}{n}}$$

with $\widehat{\mathrm{V}}_1,\widehat{\mathrm{V}}_2\in\{\mathrm{RUV},\mathrm{RDV},\mathrm{RGRV},\mathrm{RTrGRV},\mathrm{RpJV},\mathrm{RnJV}\}$,

$$\left(\eta_{\widehat{\mathrm{V}}}\right)_{i,n}=\sigma^2_{\frac{i-1}{n}}\left(f_{\widehat{\mathrm{V}}}\left((W^*)_{i,n}-W_{\frac{i-1}{n}},(W_*)_{i,n}-W_{\frac{i-1}{n}},W_{\frac{i}{n}}-W_{\frac{i-1}{n}}\right)-\frac{1}{n}\right),$$

$$\widetilde{\eta}_{\widehat{\mathrm{V}}}:=f_{\widehat{\mathrm{V}}}\left((W^*)_1,(W_*)_1,W_1\right)-1.$$

Obviously, we have

$$E\left(\widetilde{\eta}_{\widehat{\mathrm{V}}_1}\widetilde{\eta}_{\widehat{\mathrm{V}}_2}\right)\sum_{i=1}^{\lfloor nt\rfloor}\frac{1}{n}\sigma^4_{\frac{i-1}{n}}\xrightarrow{P}E\left(\widetilde{\eta}_{\widehat{\mathrm{V}}_1}\widetilde{\eta}_{\widehat{\mathrm{V}}_2}\right)\int_0^t\sigma_s^4ds,$$

so that in order to prove (10), we only have to show that the components of $\left(\frac{32}{9\pi}u_{\sigma^2}u'_{\sigma^2}+RR'\right)$ are given by $E\left(\widetilde{\eta}_{\widehat{\mathrm{V}}_1}\widetilde{\eta}_{\widehat{\mathrm{V}}_2}\right)$. This can be seen by simple, but tedious calculations using the fourth moments of $(W^*)_1$, $(W_*)_1$ and W_1,[19] which we omit.

3. In order to prove (11), we again make use of the scaling property of Brownian motion to compute

$$E\left(\left(\chi_{\widehat{\mathrm{V}}}\right)_{i,n}\left(W_{\frac{i}{n}}-W_{\frac{i-1}{n}}\right)|\mathcal{F}_{\frac{i-1}{n}}\right)=\frac{\sigma^2_{\frac{i-1}{n}}}{n}E\left(W_1 f_{\widehat{\mathrm{V}}}\left((W^*)_1,(W_*)_1,W_1\right)\right),$$

whose sum converges to $E\left(W_1 f_{\widehat{\mathrm{V}}}\left((W^*)_1,(W_*)_1,W_1\right)\right)\int_0^t \sigma_s^2 ds$. We have

- $E\left(W_1 f_{\mathrm{RUV}}\left((W^*)_1,(W_*)_1,W_1\right)\right)=E\left(W_1 2\left(W^*\right)_1\left(\left(W^*\right)_1-W_1\right)\right)=0$, because W and $W^*(W^*-W)$ are independent (e.g., Seshadri (1988)). The same reasoning applies to RDV.
- Concerning RGRV, RTrGRV, we also get the value 0, because these estimators are symmetric in W.
- For RpJV, RnJV, we have to compute

$$E\left(W_1\frac{1}{2}\left((W^*)_1^2+\left(W_1-(W_*)_1\right)^2\right)\right)=\frac{8}{3\sqrt{2\pi}},$$
$$E\left(W_1\frac{1}{2}\left((W_*)_1^2+\left(W_1-(W^*)_1\right)^2\right)\right)=-\frac{8}{3\sqrt{2\pi}},$$

  which can be seen e.g. by using the generating function given in Garman & Klass (1980) or by using the joint distributions of (W^*,W) and (W_*,W), which can be found in Borodin & Salminen (2002).

4. Due to the boundedness of σ, it suffices to prove (12) for $\sigma\equiv 1$, giving

$$n\sum_{\widehat{\mathrm{V}}}\left(f_{\widehat{\mathrm{V}}}\left((W^*)_{i,n}-W_{\frac{i-1}{n}},(W_*)_{i,n}-W_{\frac{i-1}{n}},W_{\frac{i}{n}}-W_{\frac{i-1}{n}}\right)-\frac{1}{n}\right)^2$$

for $|\chi_{i,n}|^2$, where the sum runs over

$$\widehat{\mathrm{V}}\in\{\mathrm{RUV},\mathrm{RDV},\mathrm{RGRV},\mathrm{RTrGRV},\mathrm{RpJV},\mathrm{RnJV}\}.$$

[19] These moments can be found in Garman & Klass (1980, p. 74).

We then have

$$\sum_{i=1}^{\lfloor nt \rfloor} E\left(|\chi_{i,n}|^2 1_{\{|\chi_{i,n}|>\varepsilon\}} | \mathcal{F}_{\frac{i-1}{n}}\right) = \frac{\lfloor nt \rfloor}{n} E\left(A\, 1_{\{A>n\varepsilon\}}\right),$$

with

$$A = \sum_{\widehat{\mathrm{V}}} \left(f_{\widehat{\mathrm{V}}}\left((W^*)_1, (W_*)_1, W_1\right) - 1\right)^2.$$

Now (12) follows easily from dominated convergence, as $E(A) < \infty$.

5. (13) can be proven by using the predictable representation property of Brownian motion, the argument being exactly the same as given by Christensen & Podolskij (2007, p. 344).

6.6 Proof of Theorem 7

As in the proofs of Theorems 1 and 6, we can, without loss of generality, assume that μ, σ, $\tilde{\mu}$, $\tilde{\sigma}$, and $\tilde{v}$ are bounded. Defining

$$\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n} := f_{\widehat{\mathrm{V}}}\left((p^*)_{i,n} - p_{\frac{i-1}{n}}, (p_*)_{i,n} - p_{\frac{i-1}{n}}, p_{\frac{i}{n}} - p_{\frac{i-1}{n}}\right),$$

the proof proceeds in two steps: using (5), we first show

$$\sqrt{n} \sum_{i=1}^{n} \left(\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n} - \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} - E\left(\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n} - \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} | \mathcal{F}_{\frac{i-1}{n}}\right)\right) \xrightarrow{P} 0,$$

while the proof will we completed by showing that

$$\sqrt{n} \sum_{i=1}^{n} E\left(\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n} - \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} | \mathcal{F}_{\frac{i-1}{n}}\right) \tag{14}$$

converges to the limit given in the theorem. To prove the first step, we obviously have that

$$\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n} - \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} - E\left(\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n} - \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} | \mathcal{F}_{\frac{i-1}{n}}\right)$$

is a zero-mean, uncorrelated sequence, so that it is enough to show that

$$n \sum_{i=1}^{n} E\left(\left(\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n} - \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} - E\left(\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n} - \left(\xi_{\widehat{\mathrm{V}}}\right)_{i,n} | \mathcal{F}_{\frac{i-1}{n}}\right)\right)^2\right) \xrightarrow{P} 0,$$

which can be done with the same methods as in the proof of Theorem 1. For the second step, we decompose σ_t, p_t for $t \in [\frac{i-1}{n}, \frac{i}{n}]$ as follows:

$$\sigma_t = \sigma_{\frac{i-1}{n}} + \tilde{\sigma}_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} dW_s + \tilde{v}_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} dB_s + R_{\sigma,t}$$

with

$$R_{\sigma,t} := \int\limits_{\frac{i-1}{n}}^{t} \tilde{\mu}_s ds + \int\limits_{\frac{i-1}{n}}^{t} (\tilde{\sigma}_s - \tilde{\sigma}_{\frac{i-1}{n}}) dW_s + \int\limits_{\frac{i-1}{n}}^{t} (\tilde{v}_s - \tilde{v}_{\frac{i-1}{n}}) dB_s,$$

$$\begin{aligned} p_t = p_{\frac{i-1}{n}} + \mu_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} ds + \sigma_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} dW_s \\ + \tilde{\sigma}_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} \int\limits_{\frac{i-1}{n}}^{s} dW_\tau dW_s + \tilde{v}_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} \int\limits_{\frac{i-1}{n}}^{s} dB_\tau dW_s + R_{p,t} \end{aligned}$$

with

$$R_{p,t} := \int\limits_{\frac{i-1}{n}}^{t} (\mu_s - \mu_{\frac{i-1}{n}}) ds + \int\limits_{\frac{i-1}{n}}^{t} R_{\sigma,s} dW_s.$$

Again using Cauchy-Schwarz, Burkholder-Davis-Gundy inequalities, and the decomposition introduced in the proof of Theorem (1), we find that in (14), we can replace $\left(\theta_{\widehat{\mathrm{V}}}\right)_{i,n}$ by

$$\left(\tilde{\theta}_{\widehat{\mathrm{V}}}\right)_{i,n} := f_{\widehat{\mathrm{V}}}\left((\tilde{p}^*)_{i,n}, (\tilde{p}_*)_{i,n}, \tilde{p}_{\frac{i}{n}} \right), \tag{15}$$

where for $t \in [\frac{i-1}{n}, \frac{i}{n}]$

$$\tilde{p}_t := \sigma_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} dW_s + \mu_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} ds + \tilde{\sigma}_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} \int\limits_{\frac{i-1}{n}}^{s} dW_\tau dW_s + \tilde{v}_{\frac{i-1}{n}} \int\limits_{\frac{i-1}{n}}^{t} \int\limits_{\frac{i-1}{n}}^{s} dB_\tau dW_s.$$

Obviously, for fixed i, n, $\sqrt{n}\,\tilde{p}$ equals on $[\frac{i-1}{n}, \frac{i}{n}]$

$$\sigma_{\frac{i-1}{n}} \int\limits_{0}^{\tilde{t}} d\widetilde{W}_s + \frac{1}{\sqrt{n}} \left(\mu_{\frac{i-1}{n}} \int\limits_{0}^{\tilde{t}} ds + \tilde{\sigma}_{\frac{i-1}{n}} \int\limits_{0}^{\tilde{t}} \int\limits_{0}^{s} d\widetilde{W}_\tau d\widetilde{W}_s + \tilde{v}_{\frac{i-1}{n}} \int\limits_{0}^{\tilde{t}} \int\limits_{0}^{s} d\widetilde{B}_\tau d\widetilde{W}_s. \right)$$

with $\tilde{t} := \frac{t-\frac{i-1}{n}}{\frac{1}{n}} \in [0,1]$ and the independent Brownian motions[20]

$$\widetilde{W} := \sqrt{n}\left(W_{\frac{i-1}{n}+\frac{\cdot}{n}} - W_{\frac{i-1}{n}}\right),\ \widetilde{B} := \sqrt{n}\left(B_{\frac{i-1}{n}+\frac{\cdot}{n}} - B_{\frac{i-1}{n}}\right).$$

We now follow the lines of Christensen & Podolskij (2007, p. 346 f) by applying their Lemma 1 (Christensen & Podolskij (2007, p. 343)) to $\varepsilon = \frac{1}{\sqrt{n}}$ and the functions

$$f_{i,n} := \sigma_{\frac{i-1}{n}} \int_0^{\cdot} d\widetilde{W}_s,$$

$$g_{i,n} := \mu_{\frac{i-1}{n}} \int_0^{\cdot} ds + \tilde{\sigma}_{\frac{i-1}{n}} \int_0^{\cdot}\int_0^{s} d\widetilde{W}_\tau d\widetilde{W}_s + \tilde{v}_{\frac{i-1}{n}} \int_0^{\cdot}\int_0^{s} d\widetilde{B}_\tau d\widetilde{W}_s,$$

defined on $[0,1]$, which gives us the decompositions

$$\sqrt{n}\left((\tilde{p}^*)_{i,n} - \sigma_{\frac{i-1}{n}}\left((W^*)_{i,n} - W_{\frac{i-1}{n}}\right)\right) = \frac{1}{\sqrt{n}}\left((g_{i,n})_{t^*(\widetilde{W})} + (R^*)_{i,n}\right),$$

$$\sqrt{n}\left((\tilde{p}_*)_{i,n} - \sigma_{\frac{i-1}{n}}\left((W_*)_{i,n} - W_{\frac{i-1}{n}}\right)\right) = \frac{1}{\sqrt{n}}\left((g_{i,n})_{t_*(\widetilde{W})} + (R_*)_{i,n}\right),$$

$$\sqrt{n}\left(\tilde{p}_{\frac{i}{n}} - \sigma_{\frac{i-1}{n}}\left(W_t - W_{\frac{i-1}{n}}\right)\right) = \frac{1}{\sqrt{n}}\,(g_{i,n})_1\,,$$

with $E\left((R^*)^2_{i,n}\right) = o_P(1) = E\left((R_*)^2_{i,n}\right)$ (uniformly in i) and

$$t^*(\widetilde{W}) := \arg\sup_{t\in[0,1]} \widetilde{W}_t,\ t_*(\widetilde{W}) := \arg\inf_{t\in[0,1]} \widetilde{W}_t.$$

Combining this with (14), (15), we have to find the limit of

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} E\left(f_{\widehat{\mathrm{V}}}\left(\sigma_{\frac{i-1}{n}}\widetilde{W}^* + \frac{1}{\sqrt{n}}a_{i,n}, \sigma_{\frac{i-1}{n}}\widetilde{W}_* + \frac{1}{\sqrt{n}}b_{i,n}, \sigma_{\frac{i-1}{n}}\widetilde{W}_1 + \frac{1}{\sqrt{n}}b_{i,n}\right)\right.$$
$$\left. - f_{\widehat{\mathrm{V}}}\left(\sigma_{\frac{i-1}{n}}\widetilde{W}^*, \sigma_{\frac{i-1}{n}}\widetilde{W}_*, \sigma_{\frac{i-1}{n}}\widetilde{W}_1\right)|\mathcal{F}_{\frac{i-1}{n}}\right),$$

where

$$a_{i,n} := (g_{i,n})_{t^*(\widetilde{W})} + (R^*)_{i,n}\,,\ b_{i,n} := (g_{i,n})_{t_*(\widetilde{W})} + (R_*)_{i,n}\,,\ c_{i,n} := (g_{i,n})_1\,,$$

$$\widetilde{W}^* := \left(\widetilde{W}^*\right)_1,\ \widetilde{W}_* := \left(\widetilde{W}_*\right)_1.$$

[20]For notational convenience, we suppress the dependence on i, n.

As $f_{\widehat{\mathrm{V}}}$ is a quadratic polynomial, decomposing it as in Theorem 1 results up to constants in sums of products of $\sigma_{\frac{i-1}{n}}\widetilde{W}$, $\frac{1}{\sqrt{n}}g_{i,n}$, and $\frac{1}{\sqrt{n}}R_{i,n}$ (possibly with * or $_*$). Using Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities once again, we see that all the sums of products including $R_{i,n}$ will vanish, as well as those where both factors include $g_{i,n}$, while the products of $\sigma_{\frac{i-1}{n}}\widetilde{W}$ cancel out. In the end, we therefore have to consider only sums of products of $\sigma_{\frac{i-1}{n}}\widetilde{W}$ and $\frac{1}{\sqrt{n}}g_{i,n}$, with the actual form depending on $\widehat{\mathrm{V}}$:

- for RUV:

$$\frac{2}{n}\sum_{i=1}^{n} E\left(\sigma_{\frac{i-1}{n}}\widetilde{W}^*\left(\left(g_{i,n}\right)_{t^*(\widetilde{W})} - \left(g_{i,n}\right)_1\right)\right.$$
$$\left. +\sigma_{\frac{i-1}{n}}\left(\widetilde{W}^* - \widetilde{W}_1\right)\left(g_{i,n}\right)_{t^*(\widetilde{W})} | \mathcal{F}_{\frac{i-1}{n}}\right)$$

- for RDV:

$$\frac{2}{n}\sum_{i=1}^{n} E\left(\sigma_{\frac{i-1}{n}}\widetilde{W}_*\left(\left(g_{i,n}\right)_{t_*(\widetilde{W})} - \left(g_{i,n}\right)_1\right)\right.$$
$$\left. +\sigma_{\frac{i-1}{n}}\left(\widetilde{W}_* - \widetilde{W}_1\right)\left(g_{i,n}\right)_{t_*(\widetilde{W})} | \mathcal{F}_{\frac{i-1}{n}}\right)$$

- for RGRV:

$$\frac{-1}{n(2\ln 2-1)}\sum_{i=1}^{n} E\left(\sigma_{\frac{i-1}{n}}\widetilde{W}^*\left(g_{i,n}\right)_{t_*(\widetilde{W})} + \sigma_{\frac{i-1}{n}}\widetilde{W}_*\left(g_{i,n}\right)_{t^*(\widetilde{W})} | \mathcal{F}_{\frac{i-1}{n}}\right)$$

- for RTrGRV:

$$\frac{1}{n(2\ln 2-1)}\sum_{i=1}^{n} E\left(\sigma_{\frac{i-1}{n}}\left(\widetilde{W}^* - \widetilde{W}_1\right)\left(\left(g_{i,n}\right)_1 - \left(g_{i,n}\right)_{t_*(\widetilde{W})}\right)\right.$$
$$\left. +\sigma_{\frac{i-1}{n}}\left(\widetilde{W}_1 - \widetilde{W}_*\right)\left(\left(g_{i,n}\right)_{t^*(\widetilde{W})} - \left(g_{i,n}\right)_1\right) | \mathcal{F}_{\frac{i-1}{n}}\right)$$

- for RpJV:

$$\frac{1}{n}\sum_{i=1}^{n} E\left(\sigma_{\frac{i-1}{n}}\widetilde{W}^*\left(g_{i,n}\right)_{t^*(\widetilde{W})}\right.$$
$$\left. +\sigma_{\frac{i-1}{n}}\left(\widetilde{W}_1 - \widetilde{W}_*\right)\left(\left(g_{i,n}\right)_1 - \left(g_{i,n}\right)_{t_*(\widetilde{W})}\right) | \mathcal{F}_{\frac{i-1}{n}}\right)$$

- for RnJV:

$$\frac{1}{n}\sum_{i=1}^{n} E\left(\sigma_{\frac{i-1}{n}}\widetilde{W}_*\left(g_{i,n}\right)_{t_*(\widetilde{W})}\right.$$
$$\left.+\sigma_{\frac{i-1}{n}}\left(\widetilde{W}^*-\widetilde{W}_1\right)\left(\left(g_{i,n}\right)_{t^*(\widetilde{W})}-\left(g_{i,n}\right)_1\right)|\mathcal{F}_{\frac{i-1}{n}}\right)$$

Using $\int_0^t\int_0^s d\widetilde{W}_\tau d\widetilde{W}_s = \frac{1}{2}(\widetilde{W}_t^2 - t)$, we now decompose the process $g_{i,n}$:

$$(g_{i,n})_t = \left(\mu_{\frac{i-1}{n}} - \frac{1}{2}\sigma_{\frac{i-1}{n}}\right)t + \frac{\tilde{\sigma}_{\frac{i-1}{n}}}{2}\widetilde{W}_t^2 + \tilde{v}_{\frac{i-1}{n}}g_t^{(3)}$$

with $g_t^{(3)} := \int_0^t \widetilde{B}_s d\widetilde{W}_s$. Replacing $\widetilde{B}$ by $-\widetilde{B}$ shows that $g^{(3)}$ plays no role in the sums above.

- Replacing $\widetilde{W}$ by $\widetilde{W}_{1-\cdot} - \widetilde{W}_1$ shows that for RUV, RDV the drift term of $g_{i,n}$ can be neglected.
- Replacing $\widetilde{W}$ by $-\widetilde{W}$ shows that for RGRV, RTrGRV the drift term as well as the leverage term can be neglected.

The theorem now follows easily by computing the corresponding joint moments of $\widetilde{W}^*$, $\widetilde{W}_*$, $\widetilde{W}_1$, $t^*(\widetilde{W})$, and $t_*(\widetilde{W})$.[21]

[21] The density of $(\widetilde{W}^*, t^*(\widetilde{W}))$ can be found in Borodin & Salminen (2002, p. 171).

References

Aldous, D. J. & Eagleson, G. K. (1978). On mixing and stability of limit theorems. *The Annals of Probability*, 6(2), 325–331.

Alizadeh, S., Brandt, M. W., & Diebold, F. X. (2002). Range-Based Estimation of Stochastic Volatility Models. *Journal of Business*, 57(3), 1047–1091.

Andersen, T. G. & Bollerslev, T. (1998). Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts. *International Economic Review*, 39(4), 885–905.

Andersen, T. G., Bollerslev, T., & Diebold, F. X. (2002). Parametric and Nonparametric Volatility Measurement. In Y. Aït-Sahalia & L. P. Hansen (Eds.), *Handbook of Financial Econometrics*.

Ball, C. A. & Torous, W. N. (1984). The Maximum Likelihood Estimation of Security Price Volatility: Theory, Evidence, and Application to Option Pricing. *Journal of Business*, 57(1), 97–112.

Bandi, F. & Russell, J. R. (2006). Separating microstructure noise from volatility. *Journal of Financial Economics*, 79(3), 655–692.

Barndorff-Nielsen, O. E., Graversen, S. E., Jacod, J., Podolskij, M., & Shephard, N. (2006a). A Central Limit Theorem for Realised Power and Bipower Variations of Continuous Semimartingales. In *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift* (pp. 33–68). Springer.

Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A., & Shephard, N. (2008a). Designing realised kernels to measure the ex-post variation of equity prices in the presence of noise. forthcoming in Econometrica.

Barndorff-Nielsen, O. E., Kinnebrock, S., & Shephard, N. (2008b). Measuring downside risk — realised semivariance. Working paper.

Barndorff-Nielsen, O. E. & Shephard, N. (2004). Power and Bipower Variation with Stochastic Volatility and Jumps. *Journal of Financial Econometrics*, 2(1), 1–37.

Barndorff-Nielsen, O. E. & Shephard, N. (2006). Econometrics of Testing for Jumps in Financial Econometrics Using Bipower Variation. *Journal of Financial Econometrics*, 4(1), 1–30.

Barndorff-Nielsen, O. E., Shephard, N., & Winkel, M. (2006b). Limit theorems for multipower variation in the presence of jumps. *Stochastic Processes and their Applications*, 116(5), 796–806.

Becker, M., Friedmann, R., & Klößner, S. (2007a). Intraday overreaction of stock prices. Working paper.

Becker, M., Friedmann, R., Klößner, S., & Sanddorf-Köhle, W. (2007b). A Hausman test for Brownian motion. *AStA - Advances in Statistical Analysis*, 91(1), 3–21.

Beckers, S. (1983). Variances of Security Price Returns Based on High, Low, and Closing Prices. *Journal of Business*, 55(1), 97–112.

Borodin, A. N. & Salminen, P. (2002). *Handbook of Brownian Motion - Facts and Formulae*. Probability and its Applications. Basel et al.: Birkhäuser, 2. edition.

Brandt, M. W. & Jones, C. S. (2006). Volatility Forecasting With Range-Based EGARCH Models. *Journal of Business & Economic Statistics*, 24(4), 470–486.

Brunetti, C. & Lildholdt, P. M. (2007). Time Series Modeling of Daily Log-Price Ranges for CHF/USD and USD/GBP. *Journal of Derivatives*, 15(2), 39–59.

Chen, C. W. S., Gerlach, R., & Lin, E. M. H. (2008). Volatility forecasting using threshold heteroskedastic models of the intra-day range. *Computational Statistics & Data Analysis*, 52(6), 2990–3010.

Chen, Z., Daigler, R. T., & Parhizgari, A. M. (2006). Persistence of volatility in futures markets. *The Journal of Futures Markets*, 26(6), 571–594.

Chou, R. Y. (2005a). Forecasting Financial Volatilities with Extreme Values: The Conditional Autoregressive Range (CARR) Model. *Journal of Money, Credit and Banking*, 37(3), 561–582.

Chou, R. Y. (2005b). Modeling the Asymmetry of Stock Movements Using Price Ranges. In *Econometric Analysis of Financial and Economic Time Series*, volume 20, Part A of *Advances in Econometrics* (pp. 231–257).

Christensen, K. & Podolskij, M. (2007). Realized range-based estimation of integrated variance. *Journal of Econometrics*, 141(2), 323–349.

Engle, R. F. & Gallo, G. M. (2006). A multiple indicators model for volatility using intra-daily data. *Journal of Econometrics*, 131(1), 3–27.

Gallant, A. R., Hsu, C.-T., & Tauchen, G. (1999). Using Daily Range Data to Calibrate Volatility Diffusions and Extract the Forward Integrated Variance. *The Review of Economics and Statistics*, 81(4), 617–631.

Garman, M. B. & Klass, M. J. (1980). On the Estimation of Security Price Volatilities from Historical Data. *Journal of Business*, 53(1), 67–78.

Gumbel, E. J. & Keeney, R. D. (1950). The Geometric Range for Distributions of Cauchy's Type. *The Annals of Mathematical Statistics*, 21(1), 133–137.

Huang, X. & Tauchen, G. (2005). The Relative Contribution of Jumps to Total Price Variance. *Journal of Financial Econometrics*, 3(4), 456–499.

Jacod, J. (1997). Jacod On continuous conditional Gaussian martingales and stable convergence in law. *Séminaire de probabilités*, 31, 232–246.

Jacod, J. (2008). Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Processes and their Applications*, 118(4), 517–559.

Kunitomo, N. (1992). Improving the Parkinson Method of Estimating Security Price Volatilities. *Journal of Business*, 65(2), 295–302.

Martens, M. & van Dijk, D. (2007). Measuring volatility with the realized range. *Journal of Econometrics*, 138(1), 181–207.

Parkinson, M. (1980). The Extreme Value Method for Estimating the Variance of the Rate of Return. *Journal of Business*, 53(1), 61–65.

Rényi, A. (1963). On stable sequences of events. *Sankhya Ser. A*, 25(3), 293–302.

Rogers, L. C. G. & Satchell, S. E. (1991). Estimating variance from high, low and closing prices. *The Annals of Applied Probability*, 1(4), 504–512.

Rogers, L. C. G., Satchell, S. E., & Yoon, Y. (1994). Estimating the volatility of stock prices: A comparison of methods that use high and low prices. *Applied Financial Economics*, 4(3), 241–247.

Seshadri, V. (1988). Exponential models, Brownian motion and independence. *The Canadian Journal of Statistics*, 16(3), 209–221.

Shiryaev, A. N. (1999). *Essentials of Stochastic Finance: Facts, Models, Theory*, volume 3 of *Advanced Series on Statistical Science & Applied Probability*. Singapur et al.: World Scientific.

Shu, J. & Zhang, J. E. (2006). Testing range estimators of historical volatility. *The Journal of Futures Markets*, 26(3), 297–313.

Taylor, S. J. (1987). Forecasting the volatility of currency exchange rates. *International Journal of Forecasting*, 3(1), 159–170.

Taylor, S. J. (2005). *Asset Price Dynamics, Volatility, and Prediction*. Princeton, Oxford: Princeton University Press.

Veraart, A. E. D. (2008). Inference for the jump part of quadratic variation of Itô semimartingales. CREATES Research Paper 2008-17.

Wiggins, J. B. (1991). Empirical Tests of the Bias and Efficiency of the Extreme-Value Variance Estimator for Common Stocks. *Journal of Business*, 64(3), 417–432.

Yang, D. & Zhang, Q. (2000). Drift-Independent Volatility Estimation Based on High, Low, Open, and Close Prices. *Journal of Business*, 73(3), 477–491.

Zhang, L. (2006). Efficient estimation of stochastic volatility using noisy observations: a multi-scale approach. *Bernoulli*, 12, 1019–1043.

Zhang, L., Mykland, P. A., & Aït-Sahalia, Y. (2005). A Tale of Two Time Scales: Determining Integrated Volatility With Noisy High-Frequency Data. *Journal of the American Statistical Association*, 100(472), 1394–1411.