The Geometry of the State Space, Correlated Voting of Independent Voters, and Voting Power

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Abstract

This paper addresses a source of correlation not discussed in the literature so far: the geometry of the state space for whose description we develop a new concept named perfect balancedness. We show that the imbalance of the state space corresponds to the correlation of the votes of independent voters which takes positive values for non-perfectly balanced state spaces and vanishes for perfectly balanced ones.

Next, we show how various measures for success, luck, and decisiveness react to positive correlation. We then define three measures of power and show that they all agree with the Banzhaf in the absence of correlation. For correlation converging to unity, they all approach a newly defined power index, named 'extreme correlation power index'. Finally, we do some comparative statics w.r.t. the voting rules and find, in particular, that a powerless voter is best off when the others decide according to the simple majority rule.

Keywords: correlated votes, spatial voting, success, decisiveness, luck, power indices

JEL classification: D70, D72, C70

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1 Introduction

In the literature, several explanations for correlated voting have been discussed: common or opposite preferences, strategic behavior, tacit collusion between members of a voting body, and common beliefs or standards (Kaniovski 2008a, Kaniovski 2008b). In this paper we deal with a further source of correlation: the geometry of the state space. We develop a new concept for the description of the properties of a state space, labeled perfect balancedness. We show that a perfectly balanced state space is necessary and sufficient for non-correlation of the votes of independent voters. If a state space is not perfectly balanced the votes of independent voters are necessarily positively correlated – which is an unplanned outcome of social interaction.

It is obvious that the answer to the question of whether or not votes are correlated may have far reaching consequences for the choice of the 'right' power index in empirical work. Empirical evidence suggests that positive correlation of votes is a widespread phenomenon. It could be identified for judicial bodies, such as the U.S. Supreme Court (Kaniovski & Leech 2009) and the Supreme Court of Canada (Heard & Swartz 1998), as well as non-judicial bodies, such as the Council of Ministers of the European Union (Hayes-Renshaw, van Aken & Wallace 2006) and the United Nations (Newcombe, Ross & Newcombe 1970). Positive correlation could also be observed in general elections (Gelman, Katz & Boscardin 1998, Gelman, Katz & Bafumi 2004).

To demonstrate the important role played by correlated voting in social choice and particularly in collective bodies that make yes-or-no decisions by vote (juries, councils, legislatures, committees, or a shareholders' meeting and even nation states participating in a referendum), we analyze the sensitivity of power measures to the geometry of the state space. The literature on voting power did not address this problem – with one exception: in 2001, Felsenthal and Machover formulated a proposition that the non-normalized Strategic Power Index is simply the non-normalized Banzhaf power index multiplied by a constant that depends on the geometry of the state space (Felsenthal & Machover 2001, p. 95), which is equivalent to the normalized versions agreeing.

The Banzhaf (Banzhaf 1965, Dubey & Shapley 1979) – originally invented by the statistician Penrose (1946), Penrose (1952) – indicates the number of times each voter in a collective body that makes yes-or-no decisions by vote is critical (decisive, pivotal, has a swing), i.e. may turn a losing coalition into a winning coalition, and vice versa. It only includes in its framework the set of players, their voting weights, and the decision-making rule and serves to measure the distribution of a priori or ex ante voting power without having any recourse to a state space or to other factors present in a particular voting environment. The Banzhaf can be derived as a cooperative game value or, alternatively, interpreted as a probability of some pivotal position (see Straffin (1977), Straffin (1988)). In its probabilistic version it is based on two assumptions (Felsenthal & Machover 1998, p. 37): (1) voters cast their votes independently of all other members of a voting body and (2) each voter is equally likely to vote 'yes' or 'no' on randomly chosen proposals.

The Strategic Power Index (SPI) employs, in contrast to the Banzhaf and other commonly used power indices, the analytical tools of non-cooperative game theory. It integrates actor preferences, the state space (often referred to as a policy or issue space), as well as the rules of the decision-making process, into the analysis. This index, which has been developed in several papers[1], uses the expected or average distance between players' ideal points and the equilibrium outcome in policy games in its power calculus. The smaller the expected distance, the more power is attributed to a player compared to a 'neutral' or dummy player who does not have any decision-making rights in the game.[2]

Felsenthal and Machover derived their interesting and, given the rather distinctive analytics characterizing the Banzhaf and the Strategic Power Index, surprising result for state spaces that they called 'perfectly symmetric'. However, Felsenthal and Machover did not address the problem of state space generated correlation. In this paper we discuss the problem of the interplay of the geometry of the state space and voting power from a much more fundamental and broader perspective.

First of all, with the concept of a perfectly balanced state space we deliver a precise base for the description of the properties of a state space: to this end, consider a spatial voting game and three random variables defined on the state space, representing a proposal, the status quo and the ideal point of a voter. The state space is perfectly balanced if, with probability 1, the status quo and the proposal are such that the probability of a random ideal point being closer to the proposal equals that of being closer to the status quo. We develop a measure for the imbalance of the state space and show that the higher the overall imbalance the higher the correlation. We exemplify the above results by looking at a few specific state spaces: one-dimensional state spaces (a compact set and the whole line of real numbers) and the two-dimensional state spaces represented by the vertices, edges, and area of a rectangle and the line and area of a circle.

Secondly, we define probability-based measures for success, luck, and decisiveness as well as a measure for the average distance between players' ideal points and the outcome of the vote. Success means that a player's vote coincides with the outcome of a voting procedure – acceptance or rejection, irrespective of whether his/her vote was crucial for it or not; a player is lucky if he/she is successful without being decisive, with decisiveness defined as obtaining the result one voted for and being pivotal (crucial) for the outcome. We show that, in the absence of correlation, these measures which are often referred to in the context of the measurement of power, only depend on the number of times the players are decisive. Only the average distance and the reduction thereof depend on the state space. All probability-based variables do not change when one perfectly balanced state space is replaced by another perfectly balanced state space.

Under positive correlation matters are much more complex: all probability- and distance-based measures then depend on the imbalance of the state space. While, for the

[1]See Steunenberg, Schmidtchen & Koboldt (1996), Steunenberg, Schmidtchen & Koboldt (1997), Steunenberg, Schmidtchen & Koboldt (1999), Schmidtchen & Steunenberg (2002), Schmidtchen & Steunenberg (2013).

[2]The SPI belongs to the class of spatial voting indices which are calculated on the base of a state space and voters' preferences, the status quo and proposals defined as points in the state space. All spatial voting power indices besides the SPI focus on coalitions and the decisiveness of players for the formation of winning coalitions (Rae 1969, Owen & Shapley 1989, Passarelli & Barr 2007, Barr & Passarelli 2009, Godfrey, Grofman & Feld 2011, Benati & Vittucci Marzetti 2013). Felsenthal and Machover were not the first connecting the Banzhaf with a spatial power index. Dubey & Shapley (1979) showed that there exists an affine-linear relation between the Rae-index (Rae (1969)) and the Banzhaf. However, this only holds true if all voting configurations are assumed being equally probable (see Dubey & Shapley (1979)).

special case of uncorrelated votes, there is an affine relation between the probabilities of being successful and being decisive, for positive correlation these probabilities do no longer go hand in hand. Therefore, success then is not a function of a player's decisiveness only, but it also depends on situations where the player is dispensable or insufficient for getting a proposal accepted.

In a much-cited paper entitled 'Is it Better to be Powerful or Lucky?', Brian Barry presented the following formula: 'success = luck + decisiveness' (Barry (1980, p. 338)). In the literature, this formula has been interpreted only in probability terms. We show that it also holds true when the three terms are reformulated as distances in the state space, indicating voters' (dis-)utility.

Normalizing decisiveness, excess success, i.e. the amount by which the success probability of a player exceeds that of the dummy player, and the reduction of average distance delivers three power measures (where the reduction of average distance gives the SPI). Interestingly, normalized excess success and the SPI share an important property of (normalized) decisiveness: they only depend on those situations where a player is pivotal. We therefore call them normalized measures of decisiveness.

Using specially designed state spaces, we discuss the effect of increasing correlation on the above mentioned variables. Typically, success and luck are increasing in correlation while decisiveness decreases when correlation rises. With respect to the normalized measures of decisiveness which are sensitive to changes in correlation, we study their behavior in relation to three correlation-insensitive power measures: the Banzhaf, the Shapley-Shubik, and a newly developed index named 'extreme correlation power index' (ECPI). All normalized measures of decisiveness agree with the Banzhaf in case of zero correlation, i.e. for perfectly balanced state spaces. When correlation is close to unity, they are close to the ECPI, and when correlation is positive but moderate, they sometimes are well approximated by the Shapley-Shubik index. This is not surprising since it is known in the literature that the Shapley-Shubik index can be derived under specific probabilistic assumptions inducing correlation, corresponding to Cauchy distributed random variables in our state space setting.

Finally, we derive some comparative statics results regarding voting rules. In the absence of correlation, increasing the number of situations in which a player is pivotal is the only way for a player to be better off, both in terms of success and decisiveness. In the presence of positive correlation, this is no longer true. For instance, by changing the voting rules, players may trade decisiveness for being more successful: although losing decisiveness, a player can gain in terms of success due to the change of voting rule significantly increasing the probability of being lucky. Another new aspect coming with positive correlation is that it is possible that, by changing the voting rule, the number of constellations decreases where a certain player is decisive, while this player's probability of being decisive increases! This surprising result is due to the fact that under positive correlation, constellations are no longer equiprobable: with increasing correlation, it becomes more and more likely to encounter outcomes of the vote with a vast majority of the players voting either for or against the proposal. In other words, it becomes less and less likely that voters split into two almost equally large groups, and therefore it is less important to be decisive in these situations and more important to be decisive in situations of very unbalanced voting.

Moreover, under positive correlation, powerless players will be successful more than

50% of the time, a kind of positive externality in terms of success that players with power produce to the benefit of powerless voters. This externality is largest when the powerful voters decide according to the simple majority rule so that it is this rule that leaves powerless players best off.

A final point is worth to be mentioned: power indices, such as the Banzhaf and other traditional measures, taking the independence of voters as a necessary and sufficient condition for zero-correlation, can deliver unbiased power scores only if state spaces are perfectly balanced.[3] Unless one decides to neglect the state space issue entirely or claims that state spaces are mainly perfectly balanced, power measures are needed that deliver unbiased power scores for uncorrelated voting due to perfectly balanced state spaces as well as for correlated voting caused by imbalanced state spaces.[4]

The paper is organized as follows: Section 2 outlines the basics of our state space set-up and describes the relation between the votes' correlation and the geometry of the state space. After providing definitions and formulas for success, decisiveness, luck and power in Section 3, we deal in Section 4 with the influence of the geometry of the state space on these measures. Section 5 discusses the effect of increasing correlation on success, luck, decisiveness, and power while Section 6 is concerned with comparative statics. Section 7 concludes, additional tables as well as proofs can be found in the appendix.

2 The geometry of the state space

2.1 The set-up

We consider a voting body of n players/voters, $\{1, \ldots, n\}$, to which we add a dummy player 0 which has no influence on the voting body's decisions[5]: we denote by

$$N := \{0, \ldots, n\}$$

the set of all players, including the dummy player. The actual voters form collective decisions by voting in favor of a given proposal, p, or for staying with the status quo, q: formally, every voter i decides on whether he votes in favor of the proposal, $a_i = 1$, or against it, $a_i = 0$. The decisions of the voting body are described by a function v which assigns to every subset $M := \{i \in N : a_i = 1\} \subseteq N$ of supporters of the proposal either the value 1 (the proposal is accepted) or the value 0 (the status quo is maintained). v represents the rules underlying the voting decision, e.g. the simple majority rule or situations that require unanimity for the proposal to be accepted. We assume that the voting rule v has the following properties:

- $v(\varnothing) = 0$, i.e. if no one votes in favor of the proposal, then the proposal is dismissed,

[3] Kaniovski (2008a) identifies a substantial bias in the Banzhaf measure if votes are neither equiprobable nor independent. Kaniovski & Das (2012) show, running simulations, that the Banzhaf as well as the Shapley-Shubik measure of power tend to overestimate voting power when votes are positively correlated.

[4] As for the postulate to base power calculations on more flexible and more empirically informed distributions of votes see also Kaniovski & Das (2012), Laruelle & Valenciano (2005), Kaniovski (2008b), Kaniovski & Leech (2009).

[5] See below.

- $v(N) = 1$, i.e. in case of an unanimous vote for the proposal, it is successful,
- if $M_1 \subseteq M_2$, then $v(M_1) \leq v(M_2)$, i.e. adding supporters of the proposal cannot turn an initially successful proposal into an unsuccessful one.[6]

As stated above, player 0 is a dummy player, i.e. player 0's voting has no influence on whether the proposal p is successful or the status quo q is maintained, formally:

$$\forall M \subseteq N : v(M \setminus \{0\}) = v(M) = v(M \cup \{0\}),$$

i.e. for all possible sets M of supporters of the proposal, the body's decision v is unaffected by player 0's decision to vote in favor of or against the proposal.

The players' (including the dummy's) preferences are described by ideal points $x_0, \ldots, x_n$ which are elements of a state space[7] X such that player i strictly prefers the proposal p over the status quo if its distance to player i's ideal point x_i is smaller than that of the status quo. More formally, we assume that player i will vote in favor of the proposal, $a_i = 1$, if the proposal is closer to the ideal point than the status quo:

$$\forall\, i = 0, \ldots, n : d(p, x_i) < d(q, x_i) \;\Rightarrow\; a_i = 1.$$

Accordingly, we have

$$\forall\, i = 0, \ldots, n : d(p, x_i) > d(q, x_i) \;\Rightarrow\; a_i = 0.$$

In case of a tie, we assume that players toss a coin:

$$\forall\, i = 0, \ldots, n : d(p, x_i) = d(q, x_i) \;\Rightarrow\; a_i = 0, a_i = 1 \text{ each with probability one half}^{8}.$$

We call a player i successful if i's vote coincides with the voting body's decision:

$$a_i = v(\{j \in N : a_j = 1\}),$$

we call voter i decisive if the body's vote crucially hinges on i's vote:

$$v(\{j \in N \setminus \{i\} : a_j = 1\}) = 0 < 1 = v(\{j \in N \setminus \{i\} : a_j = 1\} \cup \{i\}),$$

and we call player i lucky if i is successful without being decisive. These notions are intimately linked to the following quantities which will prove very useful later on:

$$\forall\, i = 0, \ldots, n : \mathrm{Disp}_i := \{M \subseteq N \setminus \{i\} : v(M) = 1\},$$

the set of all situations in which player i is dispensable for the proposal to be accepted,

$$\forall\, i = 0, \ldots, n : \mathrm{Ins}_i := \{M \subseteq N \setminus \{i\} : 0 = v(M \cup \{i\})\},$$

[6] Formally, v is a simple monotonic game. Simple monotonic games constitute a very large class of voting games which in particular contains all weighted majority games.

[7] Formally, (X, d) is a metric space. Usually, the state space X will be some subset of the k-dimensional space $\mathbb{R}^k$, while d is the Euclidean distance.

[8] Other assumptions about the behavior of players in case of a tie are possible. For many state spaces (esp. non-discrete ones), results would not change at all, for others, the results of this paper would have to be slightly modified.

the set of all situations in which player i is insufficient for the proposal to be accepted, and

$$\forall\, i = 0, \dots, n : \mathrm{Piv}_i := \{M \subseteq N \setminus \{i\} : 0 = v(M) < v(M \cup \{i\}) = 1\},$$

the set of all situations in which player i's decision is pivotal for the proposal to be passed or not.[9] Obviously, player i is successful if and only if one of the following three cases occurs:

1. player i votes in favor of the proposal and the other players vote such that 'Disp$_i$ occurs', i.e. such that the subset of voters different from i and supporting the proposal is an element of Disp_i,

2. player i votes against the proposal and the other players vote such that Ins_i occurs,

3. the other players vote such that Piv_i occurs resulting in player i being pivotal and therefore successful, regardless of player i's voting decision.

In the first two of the above cases, i.e. when Ins_i or Disp_i occurs, player i is successful without being decisive, i.e. player i is lucky. In the third case, i.e. when Piv_i occurs, player i is decisive.

While Disp_i, Ins_i, and Piv_i refer to N, i.e. to all players including the dummy player, their variants

$$\widetilde{\mathrm{Disp}}_i := \{M \subseteq \{1, \dots, n\} \setminus \{i\} : v(M) = 1\},$$
$$\widetilde{\mathrm{Ins}}_i := \{M \subseteq \{1, \dots, n\} \setminus \{i\} : 0 = v(M \cup \{i\})\},$$
$$\widetilde{\mathrm{Piv}}_i := \{M \subseteq \{1, \dots, n\} \setminus \{i\} : 0 = v(M) < v(M \cup \{i\}) = 1\}$$

for $i = 1, \dots, n$ give the corresponding notions when the dummy is ignored.

We now come to the central building block of our approach: we assume that the ideal points of all players as well as the status quo and the proposal are realizations of random variables with the following properties:

1. the random ideal points $X_0, \dots, X_n$ of all players are i.i.d. random variables[10], i.e. they are stochastically independent and identically distributed random variables whose distribution we denoty by μ,

2. the pair (Q, P) of (random) status quo and proposal is stochastically independent of $X_0, \dots, X_n$, i.e. the players' ideal points are independent of both status quo and proposal,

3. to avoid trivial cases where status quo and proposal coincide, we assume that (Q, P) takes values in

   $$X^2_{\neq} := \{(q, p) : q, p \in X, q \neq p\},$$

   the set of all pairs of status quo q and proposal p which do not coincide,

[9] Notice that Piv_0 is always empty, as the dummy player is never pivotal. Similarly, in terms of cooperative game theory and ignoring the dummy player 0, Disp_0 is the set of all winning coalitions of the game v while Ins_0 gives all its losing coalitions.

[10] We will denote random variables by capital letters and their realizations by the corresponding lower-case letters.

4. the pair (Q, P) is distributed according to $\mu \times \mu$ conditional on $X^2_{\neq}$,

$$\nu(\cdot) := (\mu \times \mu)(\cdot | X^2_{\neq}),$$

i.e. status quo and proposal come from the same distribution μ as the ideal points, with the additional condition that they must not coincide.

By taking the ideal points as well as the status quo and the proposal to be voted upon as random, we take an ex ante approach: instead of analyzing a certain fixed scenario of these quantities, we will be taking expectations of different quantities such as the players' probabilities to be successful, to be decisive, or to be lucky.

We denote by

$$A_0(Q, P, X_0, \ldots, X_n), \ldots, A_n(Q, P, X_0, \ldots, X_n)$$

the random variables that describe the actions of players $0, \ldots, n$, i.e. the actions that players $0, \ldots, n$ choose given status Quo Q, proposal P, and ideal points $X_0, \ldots, X_n$, while

$$\overline{A} := \overline{A}(Q, P, X_0, \ldots, X_n) := v\left(\{i \in N : A_i(Q, P, X_0, \ldots, X_n) = 1\}\right)$$

denotes the random variable that describes the vote's result: it takes the value 1 if there are enough supporters for the proposal to be accepted, and 0 otherwise.

From the definition of $A_0, \ldots, A_n$, it is obvious that, for every player i, her action $A_i(Q, P, X_0, \ldots, X_n)$ does not depend on X_j for $j \neq i$, as $A_i(Q, P, X_0, \ldots, X_n)$ is determined completely by $d(X_i, P)$ and $d(X_i, Q)$, reflecting that the players decide independent of each other whether they want to support the proposal or not. To highlight this, we will from now on write $A_i(Q, P, X_i)$ instead of $A_i(Q, P, X_0, \ldots, X_n)$.

The above setup can be interpreted as a two-stage setup as in Leech (1990):

1. during the first stage, the status quo, q, and the proposal to be decided upon, p, are determined according to the probability measure ν. After these have been determined,

$$\pi(q, p) := \mu(\{x : d(p, x) < d(q, x)\}) + \frac{1}{2}\mu(\{x : d(p, x) = d(q, x)\})$$

is for each player the probability of her random ideal point being such that she acts in favor of the proposal, i.e. $\pi(q, p)$ is the probability of $A_i(Q, P, X_i) = 1$ conditional on $Q = q$ and $P = p$,

2. thus, in the second stage, i.e. conditional on status quo, q, and proposal, p, the random variables $A_0(q, p, X_0)$, ..., $A_n(q, p, X_n)$ are independent Bernoulli variables with probability $\pi(q, p)$ each,

3. from the definition, it is obvious that $\pi(q, p)$, the probability that a player accepts the proposal, and $\pi(p, q)$, the probability that a player acts against the proposal, sum up to unity: $\pi(q, p) + \pi(p, q) = 1$ for all q, p. As (Q, P) has the same distribution as (P, Q), this entails that overall, player i chooses to act in favor of the proposal ($A_i(Q, P, X_i) = 1$) and against the proposal ($A_i(Q, P, X_i) = 0$) with probability one half each.

4. as status quo Q and proposal P are random variables, so is $\Pi := \pi(Q, P)$: it is the random variable that determines the conditional probability of a player voting in favor of the proposal. Due to the previously stated facts, the expectation of Π is $E(\Pi) = \frac{1}{2}$.

Notice that one must be very careful when using the word 'independent' for characterizing voters' behavior. The reason is that there are at least three possibilities of what can be meant by 'independent voters':

1. the first meaning concerns the variables $A_0(Q, P, X_0), \ldots, A_n(Q, P, X_n)$: one might speak of independent voters if these random variables are stochastically independent.

2. the second meaning also concerns the above random variables, but conditional on $Q = q$ and $P = p$: one may call the voters independent if, conditional on proposal and status quo being determined, their voting decisions are stochastically independent.

3. the third meaning is the one we already alluded to above: every voter is either unaware of other voters' preferences or does not take these into account if they are known.

Whenever we speak of independent voters, we refer to the last two interpretations of independence which are always fulfilled for our setup. In contrast, whenever talking about correlated voting, this is to be understood as the unconditional correlation of $A_i(Q, P, X_i)$ and $A_j(Q, P, X_j)$ being different from zero. These meanings of independent voters and correlated voting make the following counter-intuitive statement possible:

The geometry of the state space can explain why independent voters' votes are correlated.

2.2 The relation between the votes' correlation and the geometry of the state space

The following theorem relates the correlation of the votes to the properties of the random variable $\Pi = \pi(Q, P)$.

Theorem 1. *For all players $i \neq j$, we have*

$$\mathrm{Cov}(A_i(Q, P, X_i), A_j(Q, P, X_j)) = \int\limits_{X^2_{\neq}} \pi(q, p)^2 \nu(d(q, p)) - \frac{1}{4} = \mathrm{Var}(\Pi) \geq 0,$$

$$\mathrm{Corr}(A_i(Q, P, X_i), A_j(Q, P, X_j)) = 4 \int\limits_{X^2_{\neq}} \pi(q, p)^2 \nu(d(q, p)) - 1 = 4\,\mathrm{Var}(\Pi) \geq 0.$$

Proof. See appendix B.1. □

From Theorem 1, we see that the correlation between two players' votes can never be negative. We also find that the votes will be uncorrelated if and only if the variance of Π equals 0. As the latter is only the case if Π is (almost surely) constant, votes are uncorrelated if and only if $\pi(q,p)$ equals one half for (ν-almost) all $(q,p) \in X^2_{\neq}$. Furthermore, the votes' correlation will be higher the more Π fluctuates around its mean of $\frac{1}{2}$.

In order to characterize the condition for the votes being uncorrelated, we introduce the notion of perfect balancedness of the state space.

Definition 1 (Perfectly balanced state space)**.** *The state space X or, more exactly, the distribution μ on the state space X, is called perfectly balanced if, for ν-every pair (q,p) of status quo q and proposal p:*

$$\mu(\{x : d(p,x) < d(q,x)\}) = \mu(\{x : d(q,x) < d(p,x)\}),$$

i.e. with probability 1 the status quo q and the proposal p are such that the probability of a random ideal point being closer to the given proposal p equals that of being closer to the given status quo q.

Using the definition of $\pi(q,p)$ and the fact that $\pi(q,p) + \pi(p,q) = 1$, it is very easy to see that the state space is perfectly balanced if and only if $\pi(q,p) = \frac{1}{2}$ for almost all (q,p). This immediately delivers the following two corollaries of the preceding results:

Corollary 1 (Perfect balance is equivalent to zero correlation)**.** *The state space is perfectly balanced if and only if the votes*

$$A_0(Q,P,X_0), \ldots, A_n(Q,P,X_n)$$

are mutually uncorrelated.

Corollary 2 (Imperfect balance is equivalent to positive correlation)**.** *The state space is not perfectly balanced if and only if the votes*

$$A_0(Q,P,X_0), \ldots, A_n(Q,P,X_n)$$

are mutually positively correlated.

To measure the amount of imbalance, we introduce the notions of imbalance between status quo q and proposal p,

$$\xi(q,p) := (\pi(q,p) - \pi(p,q))^2 = (2\,\pi(q,p) - 1)^2 = 4\left(\pi(q,p) - \frac{1}{2}\right)^2,$$

and overall imbalance,

$$\overline{\xi} := \int_{X^2_{\neq}} \xi(q,p)\,\nu(d(q,p)) = 4 \int_{X^2_{\neq}} \left(\pi(q,p) - \frac{1}{2}\right)^2 \nu(d(q,p)) = 4\,\mathrm{Var}(\Pi).$$

The imbalance $\xi(q,p)$ between q and p is the squared amount by which the probabilities to vote in favor of and against the proposal differ for given status quo q and proposal

p. The imbalance $\Xi := \xi(Q, P)$ is then a random variable whose expected value is the overall imbalance: $\mathrm{E}(\Xi) = \overline{\xi}$, where $\overline{\xi}$ is a measure of the imbalance of the state space as a whole. As it equals $4\,\mathrm{Var}(\Pi)$, Theorem 1 now reads as follows: the correlation between players' votes is equal to the amount of imbalance inherent in the state space which in turn is the same as the expected imbalance between status quo and proposal.

We will now exemplify the above results by looking at a few apecific state spaces[11]:

- as a first example, we consider the one-dimensional policy space, i.e. the line $X = [0, 1]$, with the metric $d(q, p) := |q - p|$ and μ being the Lebesgue measure on X such that the players' ideal points are uniformly distributed on $[0, 1]$. The probability of preferring a proposal p over a status quo q is then easily seen to be

  $$\pi(q, p) = \frac{q+p}{2} 1_{p<q} + (1 - \frac{q+p}{2}) 1_{p>q}$$

  for $q \neq p \in [0, 1]$, the imbalance between these is given by

  $$\xi(q, p) = (q + p - 1)^2,$$

  leading to an overall imbalance of $\overline{\xi} = \frac{1}{6}$ which in turn means that the votes of two players exhibit a correlation of $\mathrm{Corr}(A_i(Q, P, X_i), A_j(Q, P, X_j)) = \frac{1}{6}$.

- as a second example, we consider the whole line of real numbers, $X = \mathbb{R}$, with μ such that status quo Q, proposal P, and ideal points $X_0, \ldots, X_n$ are Cauchy distributed, i.e. μ has, w.r.t. the Lebesgue measure, density $f(x) = \frac{1}{(x^2+1)\pi}$. For the probability of preferring a proposal p over a status quo q, we get

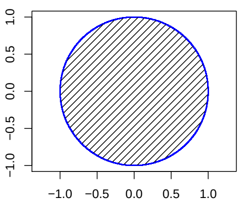
  $$\pi(q, p) = \frac{1}{2} + \frac{\arctan(\frac{q+p}{2})}{\pi} (1_{p<q} - 1_{p>q})$$

  for $q \neq p \in \mathbb{R}$, and the imbalance between q and p is given by

  $$\xi(q, p) = 4 \frac{\arctan^2(\frac{q+p}{2})}{\pi^2}.$$

  We leave it to the reader to verify that $\Pi = \pi(Q, P)$ is uniformly distributed on $[0, 1]$,[12] the corresponding density is plotted in Figure 1 (green line). This entails that $\mathrm{Var}(\Pi) = \frac{1}{12}$ and $\mathrm{Corr}(A_i(Q, P, X_i), A_j(Q, P, X_j)) = 4\frac{1}{12} = \frac{1}{3}$.

- as a third example, we consider an example of a two-dimensional state space: the circle.



[11] Results for additional examples of state spaces are available upon request.

[12] This distribution delivers the probabilistic underlying for the Shapley-Shubik index, see e.g. Leech (1990).

We discern two versions of it:

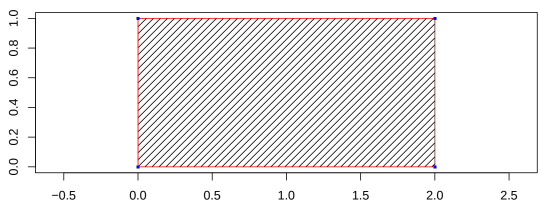
1. the circumference $X = \{(x_1, x_2) \in [-1,1]^2 : x_1^2 + x_2^2 = 1\}$, together with the uniform distribution μ on it and the Euclidean distance

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

   In this case, one finds that Π is always equal to 0.5 and we therefore have a perfectly balanced state space and zero correlation between the voters' decisions.

2. the circle's area $X = \{(x_1, x_2) \in [-1,1]^2 : x_1^2 + x_2^2 \leq 1\}$, again with the uniform distribution for μ and the Euclidean distance. In this case, it is quite tedious to calculate Π exactly, we therefore used numerical as well as simulation methods for its calculation. Figure 1 depicts the density of Π: it takes values close to one half with a higher probability as compared to the one-dimensional line as a state space, correspondingly the votes' correlation in this case is slightly lower than $\frac{1}{6}$, it is 0.123, see Table 1.

- finally, we consider three versions of a rectangle:



1. the vertices of the rectangle, $X = \{(0,0), (2,0), (2,1), (0,1)\}$, with μ assigning to each of these points the probability $\frac{1}{4}$. In this case, as for the circle's circumference, the state space is perfectly balanced and the correlations of voters' decisions are zero.

2. the edges of the rectangle, $X = \{(x, y) \in [0,2] \times [0,1] : x = 0 \vee x = 2 \vee y = 0 \vee y = 1\}$, with μ such that the ideal points are uniformly distributed on the rectangle's circumference. Due to rather complicated formulas, we again resorted so simulations: the red line in Figure 1 depicts the density of Π, showing that Π only takes values between $\frac{1}{6}$ and $\frac{5}{6}$ with values close to $\frac{1}{2}$ being quite probable. As a result, the correlation between voters' decisions takes a rather low value of 0.05, see again Table 1.

3. the area of the rectangle, $X = [0,2] \times [0,1]$, with μ the Lebesgue measure on $[0,2] \times [0,1]$ divided by the rectangle's area, 2. Again, as formulas are too complicated, we simulated all necessary quantities and present the density of Π in Figure 1 (black line). Π varies significantly, in particular also taking values values far from $\frac{1}{2}$ with considerable probability. Essentially, the density in this case lies between the corresponding values for the line and tbe circle's area. As a consequence, the votes' correlation is above that of the circle's

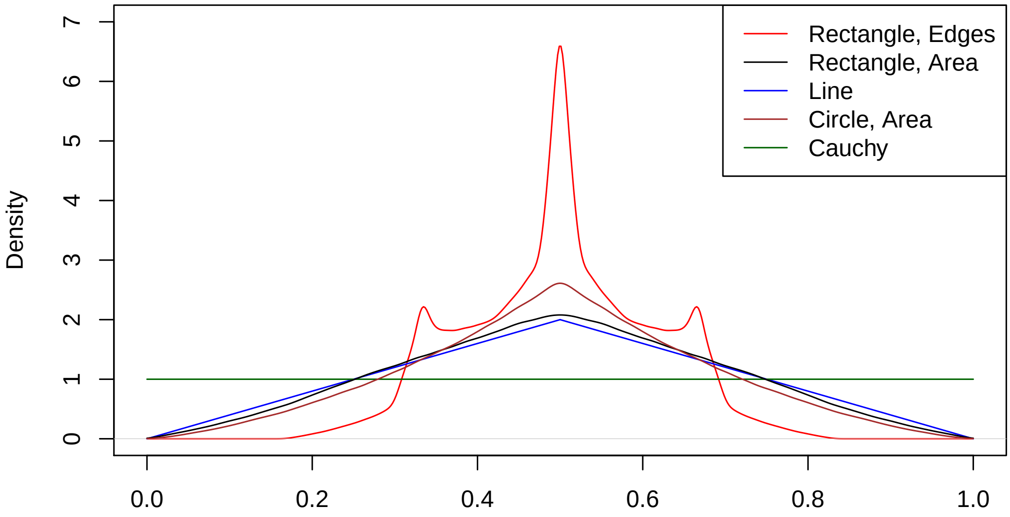


Figure 1: Density of probability Π of accepting proposal for different policy spaces

State space	Correlation
Line	$\frac{1}{6}$
Cauchy	$\frac{1}{3}$
Circle, Line	0
Circle, Area	0.123
Rectangle, Vertices	0
Rectangle, Edges	0.05
Rectangle, Area	0.148

Table 1: Correlations implied by different state spaces

area, but below that of a one-dimensional line: the exact value is 0.148, see Table 1.

2.3 Centrality

As we have seen in the previous section, correlation of the votes is linked to the geometry of the state space: the more the latter is imbalanced, the higher the correlation. In particular, the more the random variable $\Pi = \pi(Q, P)$ varies, the higher the imbalance and the votes' correlation. A nice way to visualize the geometry of the space is given

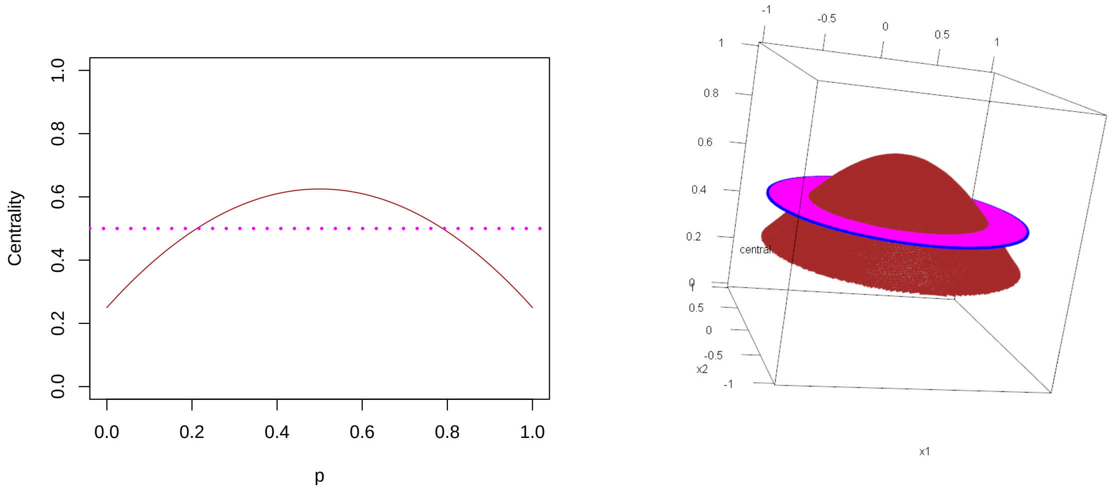


Figure 2: Centrality $\overline{\pi}$ for one-dimensional policy space: line , and two-dimensional policy space: circle

by the concept of centrality, defined as

$$\overline{\pi}(p) := \int_X \pi(q,p)\mu(\cdot|X \setminus \{p\})(dq),$$

i.e. as the probability that a player equipped with a randomly chosen ideal point will favor the given proposal p over a randomly chosen status quo. Due to its construction, $\overline{\pi}(p)$ is a measure for p's centrality: if a very central point p of X is considered as an alternative to a rather non-central or 'extreme' q, then players will have a higher probability of voting for p, $\pi(q,p) > \frac{1}{2}$, resulting in $\overline{\pi}(p) > \frac{1}{2}$ for central or 'non-extreme' points p of X and $\overline{\pi}(p) < \frac{1}{2}$ for non-central points p.

Notice that we obviously have $\overline{\pi}(p) \equiv \frac{1}{2}$ in case of a perfectly balanced state space, i.e. for perfectly balanced state spaces, there are no differences in centrality: all points $x \in X$ are equally extreme.

For imbalanced state spaces however, points have different centrality: while some are very central ($\overline{\pi}(p)$ exceeding one half), others are more or less extreme ($\overline{\pi}(p) \ll \frac{1}{2}$). For the policy spaces of the previous subsection for instance, one finds $\overline{\pi}(p) = \frac{5}{8} - \frac{3}{2}(p - \frac{1}{2})^2$ for the line $X = [0,1]$ (see left part of Figure 2): the points between $\frac{1}{2} - \frac{\sqrt{3}}{6}$ and $\frac{1}{2} + \frac{\sqrt{3}}{6}$ have above average centrality, with $\frac{1}{2}$ being the most central, and with centrality declining as points move away from $\frac{1}{2}$. While $p = \frac{1}{2}$ has a probability of 62.5% of being preferred over a randomly chosen status quo by a voter endowed with a random ideal point, this probability will diminish to only 25% for the less central, more extreme points close to 0 or 1.

For the state space with Cauchy distributed ideal points, we find that centrality is largest at points close to 0 with values of approximately 68% while the centrality of points far away from zero becomes ever smaller (see Figure 3).

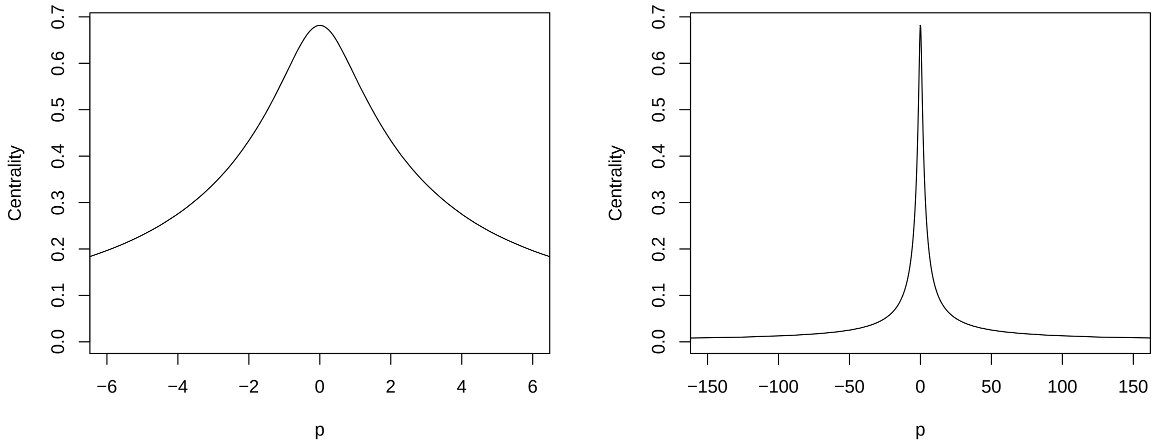


Figure 3: Centrality $\overline{\pi}$ for Cauchy distributed ideal points

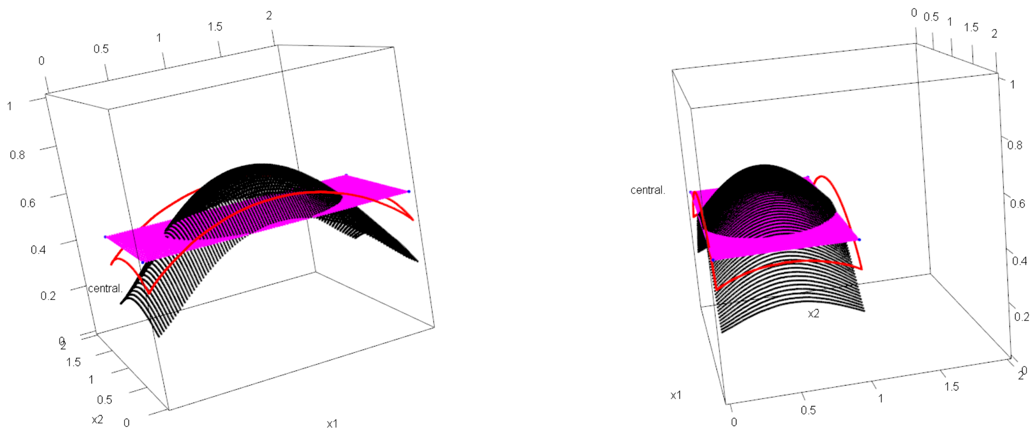


Figure 4: Centrality $\overline{\pi}$ for two-dimensional policy space: rectangle

For the circle, we find that centrality is equal to one half for the case of the circle's circumference, but varies significantly for the case of a circle's area (see right part of Figure 2): while the circle's midpoint, $(0, 0)$, has a centrality of more than 70%, centrality declines when moving from the circle's centre to its margins, taking values below 33% close to the circle's boundary.

For the rectangle, we get the following results (see Figure 4): the vertices constitute a perfectly balanced state space and therefore all have centrality one half (blue points). For the case of the rectangle's edges, the points in the middle of the longer edge have above-average centrality of almost 60% while points close to the vertices are least central with a value of about 39%. It is remarkable that all points on the shorter edge have centrality below one half, even those in the middle of the shorter edges. For the case of the area, we find the largest variation in centrality: points close to the rectangle's center at $(1, \frac{1}{2})$ are those with largest centrality of about 67% while points close to the vertices are the least central and most extreme points with centrality taking values of about 24%.

3 Success, decisiveness, luck, and power

3.1 Notions of success, decisiveness, luck, and power

In the following definition, we collect quantities that are of interest when evaluating v with respect to the players' success, desiveness, luck, and power.

Definition 2 (Success, decisiveness, luck, power)**.** *Given the above setup, we define for every player $i \in N$:*

1. *the probability of being successful, i.e. the probability of i's vote coinciding with the voting body's decision:*

$$\sigma_i' := \text{Prob}\left(A_i(Q, P, X_i) = \overline{A}(Q, P, X_0, \ldots, X_n)\right),$$

2. *the probability of being decisive, i.e. the probability of i's vote being pivotal:*

$$\delta_i' := \text{Prob}\left(v\left(M_{-i}\right) \neq v\left(\{i\} \cup M_{-i}\right)\right),$$

where $M_{-i} := \{j \in N \setminus \{i\} : A_j(Q, P, X_j) = 1\}$ denotes the (random) set of voters different from i that vote in favor of the proposal,

3. *the probability of i being lucky, i.e. the probability of being successful without being pivotal:*

$$\lambda_i := \sigma_i' - \delta_i',$$

4. *the average distance between player i's ideal point X_i and the outcome of the vote, $O := \overline{A}(Q, P, X_0, \ldots, X_n)P + \left(1 - \overline{A}(Q, P, X_0, \ldots, X_n)\right)Q$:*

$$\overline{d}_i := E\left(d\left(X_i, O\right)\right),$$

5. *the excess success, i.e. the amount by which the probability of player i being successful exceeds that of the dummy player:*

$$\sigma_i^E := \sigma_i' - \sigma_0',$$

6. *the reduction of average distance of player i as compared to the dummy player:*

$$\Psi_i' := \overline{d}_0 - \overline{d}_i,$$

7. normalized decisiveness:

$$\delta_i := \frac{\delta_i'}{\sum_{j\in N} \delta_j'} = \frac{\delta_i' - \delta_0'}{\sum_{j\in N} (\delta_j' - \delta_0')},^{13}$$

8. normalized excess success:

$$\sigma_i := \frac{\sigma_i^E}{\sum_{j\in N} \sigma_j^E} = \frac{\sigma_i' - \sigma_0'}{\sum_{j\in N} (\sigma_j' - \sigma_0')},$$

9. normalized reduction of average distance:

$$\Psi_i := \frac{\Psi_i'}{\sum_{j\in N} \Psi_j'} = \frac{\overline{d}_0 - \overline{d}_i}{\sum_{j\in N} (\overline{d}_0 - \overline{d}_j)}.$$

As powerless players, e.g. the dummy, may have significantly positive values of success[14], σ' is not a measure of power, but of success. In contrast, δ' measures power, not success: $\delta_0' = 0$, i.e. dummy players are never decisive although they are occasionally successful: sometimes, they are lucky. Luck is measured by λ which is the difference between success and decisiveness[15]: $\lambda = \sigma' - \delta'$.

Another possibility of measuring success is given by $\overline{d}$: it measures the expected distance between the outcome of the vote and the players' ideal points. As this distance can be understood as an expected disutility, we can say that player i is more successful than player j if player i's expected disutility, $\overline{d}_i$, is smaller than that of player j, $\overline{d}_j$.

In fact, one may write all the measures discussed above as expected utilities: by defining

$$u_i^\sigma(\ldots) := 1_{A_i(Q,P,X_i)=\overline{A}(Q,P,X_0,\ldots,X_n)},$$

player i's probability of being successful, σ_i', is an expected utility: $\sigma_i' = E(u_i^\sigma(\ldots))$; similarly, defining

$$u_i^\delta(\ldots) := 1_{v(M_{-i})\neq v(\{i\}\cup M_{-i})},$$

player i's probability of being decisive, δ_i', is also an expected utility: $\delta_i' = E(u_i^\delta(\ldots))$.

It is possible to transform a measure of success into a measure of decisiveness: by relating the measure of success of player i (e.g., σ_i' or $\overline{d}_i$) to the corresponding measure for the dummy player (σ_0' or $\overline{d}_0$), one can construct quantities which take the value zero for powerless or dummy players: excess success $\sigma_i^E = \sigma_i' - \sigma_0'$ and reduction of average distance $\Psi_i' = \overline{d}_0 - \overline{d}_i$. As we will see below, these are actually also measures of decisiveness as for their calculation the only relevant situations are those where player i is pivotal, i.e. the sets Piv_i.

Altogether, we therefore have three measures of decisiveness: decisiveness, δ', excess success, σ^E, and reduction of average distance, Ψ'. Each of these can be normalized to get a power index, i.e. they can be normalized such that the sum over all players equals

[13] As player 0 is a dummy player, player 0 is never decisive: $\delta_0' = 0$.

[14] In fact, one may prove that $\sigma_0' \geq \frac{1}{2}$ in our setting.

[15] See (Barry 1980).

unity. Thereby, we get three power indices: normalized decisiveness, δ, normalized excess success, σ, and normalized reduction of average distance, Ψ. Among these, normalized decisiveness δ' is by far the most prominent one in the literature. Notice, however, that Ψ' is also known under the name 'strategic power index' which is defined in much more general contexts of non-cooperative game theory.

3.2 Formulas for success, decisiveness, luck, and power

The following theorem is the central result of this section:

Theorem 2. *We have the following formulas for each $i \in N$:*[16]

1. *for the probability of being successful:*

$$\sigma_i' = \sum_{M \in \mathrm{Disp}_i} \int_{X_{\neq}^2} \pi(q,p)^{|M|+1}(1-\pi(q,p))^{n-|M|}\nu(d(q,p))$$
$$+ \sum_{M \in \mathrm{Ins}_i} \int_{X_{\neq}^2} \pi(q,p)^{|M|}(1-\pi(q,p))^{n+1-|M|}\nu(d(q,p))$$
$$+ \sum_{M \in \mathrm{Piv}_i} \int_{X_{\neq}^2} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-|M|}\nu(d(q,p))$$

2. *for the probability of being decisive:*

$$\delta_i' = \sum_{M \in \mathrm{Piv}_i} \int_{X_{\neq}^2} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-|M|}\nu(d(q,p))$$

3. *for the probability of being lucky:*

$$\lambda_i = \sum_{M \in \mathrm{Disp}_i} \int_{X_{\neq}^2} \pi(q,p)^{|M|+1}(1-\pi(q,p))^{n-|M|}\nu(d(q,p))$$
$$+ \sum_{M \in \mathrm{Ins}_i} \int_{X_{\neq}^2} \pi(q,p)^{|M|}(1-\pi(q,p))^{n+1-|M|}\nu(d(q,p))$$

[16] All results can be written down in two ways: first, including the dummy player, i.e. using $N = \{0,\ldots,n\}$ and Disp_i, Ins_i, Piv_i, and, second, excluding the dummy player, i.e. using $\widetilde{\{1,\ldots,n\}}$ and $\widetilde{\mathrm{Disp}}_i$, $\widetilde{\mathrm{Ins}}_i$, $\widetilde{\mathrm{Piv}}_i$. Notice that n in formulas of the first kind corresponds to $n-1$ in formulas of the second kind. In the text, we decided to always present the more intuitive version.

4. *for the average distance between outcome and ideal point:*

$$\overline{d}_i = \sum_{M \in \mathrm{Piv}_i} \int_{X^2_{\neq}} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-|M|} \int_X \min(d(q,x_i), d(p,x_i))\mu(dx_i)\nu(d(q,p))$$
$$+ \sum_{M \in \mathrm{Disp}_i} \int_{X^2_{\neq}} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-|M|} \int_X d(p,x_i)\mu(dx_i)\nu(d(q,p))$$
$$+ \sum_{M \in \mathrm{Ins}_i} \int_{X^2_{\neq}} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-|M|} \int_X d(q,x_i)\mu(dx_i)\nu(d(q,p))$$

5. *for the excess success:*

$$\sigma_i^E = \sigma_i' - \sigma_0' = 2 \sum_{M \in \widetilde{\mathrm{Piv}}_i} \int_{X^2_{\neq}} \pi(q,p)^{|M|+1}(1-\pi(q,p))^{n-|M|}\nu(d(q,p))$$

6. *for the reduction of average distance:*

$$\Psi_i' = \sum_{M \in \widetilde{\mathrm{Piv}}_i} \int_{X^2_{\neq}} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-1-|M|} z(q,p)\nu(d(q,p))$$

with the function

$$z(q,p) := \int_X \left(\pi(q,p)d(p,x) + (1-\pi(q,p))d(q,x) - \min(d(q,x), d(p,x))\right)\mu(dx)$$

7. *for normalized decisiveness:*

$$\delta_i = \frac{\sum_{M \in \widetilde{\mathrm{Piv}}_i} \int_{X^2_{\neq}} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-1-|M|}\nu(d(q,p))}{\sum_{j \in \{1,\ldots,n\}} \sum_{M \in \widetilde{\mathrm{Piv}}_j} \int_{X^2_{\neq}} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-1-|M|}\nu(d(q,p))}$$

8. *for normalized excess success:*

$$\sigma_i = \frac{\sum_{M \in \widetilde{\mathrm{Piv}}_i} \int_{X^2_{\neq}} \pi(q,p)^{|M|+1}(1-\pi(q,p))^{n-|M|}\nu(d(q,p))}{\sum_{j \in \{1,\ldots,n\}} \sum_{M \in \widetilde{\mathrm{Piv}}_j} \int_{X^2_{\neq}} \pi(q,p)^{|M|+1}(1-\pi(q,p))^{n-|M|})\nu(d(q,p))}$$

9. *for normalized reduction of average distance:*

$$\Psi_i = \frac{\sum_{M \in \widetilde{\mathrm{Piv}}_i} \int_{X^2_{\neq}} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-1-|M|} z(q,p)\nu(d(q,p))}{\sum_{j \in \{1,\ldots,n\}} \sum_{M \in \widetilde{\mathrm{Piv}}_j} \int_{X^2_{\neq}} \pi(q,p)^{|M|}(1-\pi(q,p))^{n-1-|M|} z(q,p)\nu(d(q,p))}$$

Instead of a formal proof, we will explain how the formulas of the preceding theorem are obtained, thereby also shedding some light on their interpretation.

For the probability of player i being successful, σ_i', we have a formula consisting of three parts:

1. a sum over all the situations where player i is dispensable with the corresponding probability of i as well as all the $|M|$ members of M voting 'Yes.' while all the $n - |M|$ players not belonging to M vote 'No.',
2. a sum over all the situations where player i is insufficient with the corresponding probability of i as well all the $n - |M|$ players not belonging to M voting 'No.' while all the $|M|$ members of M vote 'Yes.',
3. a sum over all the situations where player i's vote is pivotal with the corresponding probability of all the $n - |M|$ players not belonging to M voting 'No.' while all the $|M|$ members of M vote 'Yes.'.[17]

While the third sum obviously gives the probability of player i being decisive, the first two sums taken together give the probability of player i being lucky.

The formula for the expected distance between outcome and ideal point of player i also consists of three sums: when i is pivotal, i.e. when the other players vote such that Piv_i occurs, then player i will be able to choose between the status quo, q, and the proposal, p, and will of course vote such that the smaller of the distances $d(q, x_i)$ and $d(p, x_i)$ obtains. If, however, Disp_i occurs, then the proposal will be accepted and player i will be facing the distance $d(p, x_i)$. Finally, if Ins_i occurs and the proposal will be dismissed, the corresponding distance will equal $d(q, x_i)$.

For the excess success, the situations where player i is dispensable or insufficient play no role at all: if player i lacks power to change the result of the vote, then so does the dummy player such that the corresponding parts in σ_i' and σ_0' cancel out each other. Therefore, we have to consider only those situations in which player i is decisive and therefore also successful. In those situations, the dummy player will be unsuccessful exactly if players i and the dummy player cast different votes. The conditional probability of their votes being different is $2\pi(q, p)(1 - \pi(q, p))$ which, to give the formula of the theorem, is multiplied by $\pi(q, p)^{|M|}(1 - \pi(q, p))^{n-1-|M|}$, the probability that the corresponding situation occurs.

With regard to the reduction of average distance, the situations where player i is not decisive play no role at all, for exactly the same reason as above: when player i is not decisive, there is no difference to a powerless dummy player. We therefore only have to consider the sets $\widetilde{\text{Piv}}_i$ where all other players (excluding the dummy player) vote such that player i is decisive. In such a situation, the function

$$z(q, p) = \int_X \left(\pi(q, p) d(p, x) + (1 - \pi(q, p)) d(q, x) - \min(d(q, x), d(p, x))\right) \mu(dx).$$

describes the expected advantage in terms of distance of player i over a dummy player: the result of the vote will with conditional probability $\pi(q, p)$ (the conditional probability of player i voting 'Yes.') be equal to the proposal p, and with conditional probability $1 - \pi(q, p)$ (the conditional probability of player i voting 'No.') will it be equal

[17] Player i is always successful when Piv_i occurs, regardless of how player i actually votes.

to the status quo q. Therefore, conditional on the situation, the expected distance between the dummy player's ideal point and the outcome is $\int_X \pi(q,p)d(p,x_0) + (1 - \pi(q,p))d(q,x_0)\mu(dx_0)$ while the expected distance between the outcome and the ideal point of player i is $\int_X \min(d(q,x_i), d(p,x_i))\mu(dx_i)$ because i always votes such that the smaller of the distances $d(q,x_i)$, $d(p,x_i)$ prevails. In the end, the reduction of average distance is given by multiplying $z(q,p)$ by the probability of M occurring, i.e. by $\pi(q,p)^{|M|}(1-\pi(q,p))^{n-1-|M|}$, and integrating over status quo and proposal, (q,p).

An important aspect of these formulas already alluded to above is that these quantities only depend on what happens in the situations where the players are decisive. Keeping in mind that the dummy's measure of success is a proxy for luck, we might phrase this aspect as the following generalization of Barry's formula 'decisiveness = success - luck':[18]

'Relating a measure of success to the corresponding measure of a dummy player delivers a measure of decisiveness.'

In the literature, Barry's formula has been interpreted only in probability terms. Our approach allows reformulating it in terms of distances in the state space as well. Two options exist, both indicating player i's excess success with respect to the dummy player:

$$\overline{d}_i = \overline{d}_0 - \Psi'_i$$

$$-\overline{d}_i = -\overline{d}_0 + \Psi'_i$$

Player i's average distance reflects the influence of being lucky and decisive, an influence captured on the one hand by the purely luck-driven average distance of the dummy player, and, on the other hand, a term indicating the difference in both players' average success which is the result of i's decisiveness.

Recall that expected distance between player i's ideal point and the outcome of the vote can be interpreted as expected disutility, which gives another interpretation of the first of the above mentioned formulas: player i's expected disutility equals the dummy player's purely luck-driven expected disutility corrected by the utility gain player i can achieve by being decisive now and then. The second formula expresses the same logic, however, in terms of utility.

4 The influence of correlated voting on success, decisiveness, luck, and power

4.1 Formulas in the absence of correlation

We now discuss the special case of a perfectly balanced state space, i.e. the case of zero correlation.

[18]For an extensive formal discussion of the relationship between success and decisiveness see Laruelle & Valenciano (2005), Laruelle, Martinez & Valenciano (2006), Laruelle & Valenciano (2008).

Corollary 3 (Formulas for perfectly balanced state spaces)**.** *If the state space is perfectly balanced, we have the following formulas:*[19]

1. $\sigma_i' = \frac{1}{2} + \left(\frac{1}{2}\right)^{n+1} |\operatorname{Piv}_i|$,

2. $\delta_i' = \left(\frac{1}{2}\right)^{n} |\operatorname{Piv}_i|$,

3. $\lambda_i = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1} |\operatorname{Piv}_i|$,

4. $\overline{d}_i = E\left(d\left(Q, X_i\right)\right) - \left(\frac{1}{2}\right)^{n} |\operatorname{Piv}_i| \big(E\left(d\left(Q, X_i\right)\right) - E\left(\min\left(d\left(Q, X_i\right), d\left(P, X_i\right)\right)\right)\big)$
$= E\left(d\left(P, X_i\right)\right) - \left(\frac{1}{2}\right)^{n} |\operatorname{Piv}_i| \big(E\left(d\left(P, X_i\right)\right) - E\left(\min\left(d\left(Q, X_i\right), d\left(P, X_i\right)\right)\right)\big)$,

5. $\sigma_i^E = \left(\frac{1}{2}\right)^{n} |\widetilde{\operatorname{Piv}_i}|$

6. $\Psi_i' = \left(\frac{1}{2}\right)^{n-1} |\widetilde{\operatorname{Piv}_i}| \big(E\left(d\left(Q, X_i\right)\right) - E\left(\min\left(d\left(Q, X_i\right), d\left(P, X_i\right)\right)\right)\big)$
$= \left(\frac{1}{2}\right)^{n-1} |\widetilde{\operatorname{Piv}_i}| \big(E\left(d\left(P, X_i\right)\right) - E\left(\min\left(d\left(Q, X_i\right), d\left(P, X_i\right)\right)\right)\big)$,

7. $\delta_i = \sigma_i = \Psi_i = \dfrac{|\widetilde{\operatorname{Piv}_i}|}{\sum\limits_{j \in \{1,\dots,n\}} |\operatorname{Piv}_j|} =: \beta_i.$

A first striking observation concerning the formulas of the preceding corollary is that all the above formulas are independent of Ins_i and Disp_i, they only depend on $|\operatorname{Piv}_i|$, the number of times the players are decisive. In particular, all measures of success, decisiveness, or power for player i are increasing in the number of times player i is decisive. Therefore, from a constitutional point of view, in order to optimize any one of these quantities, players have the incentive to change the voting rules such that there are more constellations in which they are decisive.

The formula for luck, λ_i, is somewhat peculiar: it is the only one that is decreasing in $|\operatorname{Piv}_i|$. This is no surprise since increasing $|\operatorname{Piv}_i|$ leads to i being decisive more often, and when being decisive a player is not lucky, simply because he does not need luck to get the desired outcome of the vote.

Another striking result of the preceding corollary is that all the power indices, i.e. all the normalized measures of decisiveness, take the same value $\delta_i = \sigma_i = \Psi_i = \frac{|\widetilde{\operatorname{Piv}_i}|}{\sum_{j \in \{1,\dots,n\}} |\operatorname{Piv}_j|} = \beta_i$ which is just the well-known Banzhaf measure of power. In other words: in the absence of correlation, all the normalized measures of decisiveness are just Banzhaf indices in disguise. In particular, Corollary 3 is a generalization of a result of Felsenthal & Machover (2001) which states that the Banzhaf index equals the strategic power index when the state space is perfectly symmetric.

Additionally, in the absence of correlation, only the average distance and the reduction thereof depend on the state space. All other quantities do not change at all when one perfectly balanced state space is replaced by another perfectly balanced one. This can for instance be verified from Tables 2 and 11 or Tables 5 and 14 which display the

[19] The proof follows easily from the preceding theorem, occasionally using that the disjoint union of Disp_i, Piv_i, and Ins_i is simply the set of all subsets of $N \setminus \{i\}$ which contains 2^n elements, and that Q and P are identically distributed.

measures for the the circumference of a circle and the area of a rectangle as state spaces and for two voting situations of six players with voting weights 2, 2, 21, 28, 31, and 31, and a quorum of 59.5 or 86.25 votes necessary for a proposal to be accepted.[20]

4.2 Formulas for positively correlated votes

In order to discuss and interpret the changes that come along with (positive) correlation, the following corollary which is essentially a reformulation of Theorem 2 will prove useful.

Corollary 4 (Formulas for imbalanced state spaces)**.**

1. *For non-negative integers j, k, l, we have for*

$$w_{j,k,l} := E\left(\pi(Q,P)^j\,(1-\pi(Q,P))^k\,z(Q,P)^l\right):$$

$$w_{j,k,l} = E\left(\frac{z(Q,P)^l}{2^{j+k+1}}\left(\left(1-\sqrt{\xi(Q,P)}\right)^j\left(1+\sqrt{\xi(Q,P)}\right)^k + \left(1-\sqrt{\xi(Q,P)}\right)^k\left(1+\sqrt{\xi(Q,P)}\right)^j\right)\right),$$

2. $\sigma_i' = \sum_{M\in\mathrm{Disp}_i} w_{|M|+1,n-|M|,0} + \sum_{M\in\mathrm{Ins}_i} w_{|M|,n+1-|M|,0} + \sum_{M\in\mathrm{Piv}_i} w_{|M|,n-|M|,0}$

3. $\delta_i' = \sum_{M\in\mathrm{Piv}_i} w_{|M|,n-|M|,0}$

4. $\lambda_i = \sum_{M\in\mathrm{Disp}_i} w_{|M|+1,n-|M|,0} + \sum_{M\in\mathrm{Ins}_i} w_{|M|,n+1-|M|,0}$

5. $\sigma_i^E = 2\sum_{M\in\widetilde{\mathrm{Piv}}_i} w_{|M|+1,n-|M|,0}$

6. $\Psi_i = \sum_{M\in\widetilde{\mathrm{Piv}}_i} w_{|M|,n-1-|M|,1}$

7. $$\delta_i = \frac{\sum_{M\in\widetilde{\mathrm{Piv}}_i} w_{|M|,n-1-|M|,0}}{\sum_{j\in\{1,\ldots,n\}}\sum_{M\in\widetilde{\mathrm{Piv}}_j} w_{|M|,n-1-|M|,0}}$$

8. $$\sigma_i = \frac{\sum_{M\in\widetilde{\mathrm{Piv}}_i} w_{|M|+1,n-|M|,0}}{\sum_{j\in\{1,\ldots,n\}}\sum_{M\in\widetilde{\mathrm{Piv}}_j} w_{|M|+1,n-|M|,0}}$$

[20] The pattern of voting rights represents the assignment of voting rights to the 6 districts of Nassau County, NY, which reflected proportionally the respective population (with a majority of 58 votes needed to carry a motion). John F. Banzhaf III. argued in a law suit that this assignment of the 115 voting rights is seriously flawed, since three districts (North Hampstead, Oyster Bay and Glen Clove), with, respectively, 21, 2 and 2 votes, are actually powerless.

9. $\Psi_i = \dfrac{\sum\limits_{M\in\widetilde{\mathrm{Piv}}_i} w_{|M|,n-1-|M|,1}}{\sum\limits_{j\in\{1,\ldots,n\}} \sum\limits_{M\in\widetilde{\mathrm{Piv}}_j} w_{|M|,n-1-|M|,1}}$

Proof. For the proof of the first assertion, we refer to appendix B.2. The remaining assertions are easy consequences of the first and Theorem 2. □

From the preceding corollary, we learn that all probability-based measures like success, decisiveness, and luck, only depend on the state space through the distribution of certain moments of $\xi(Q, P)$ through $w_{\cdot,\cdot,0}$. For the distance-based measures, we find that they additionally depend on $z(Q, P)$, the random variable which describes the advantage in terms of reduction of average distance a player has over the dummy player in situations where the player is decisive.

Compared to the case of zero correlation, luck as well as measures of success do now additionally depend on the situations in which a player is insufficient or dispensable. In other words: it is possible to have two different voting situations with identical Piv_i, i.e. identical sets of situations where player i is decisive, but different measures of success due to differing sets of situations where the player is dispensable or insufficient. For instance, comparing Tables 4 and 7 reveals that the dummy player (which is never decisive) is better off in terms of luck, success, and average distance under the voting rule of the first table and would therefore certainly prefer that voting rule to the one underlying the second table.[21]

Measures of decisiveness as well as the power indices, however, depend only on those situations where a player is pivotal. As $w_{|M|,n-|M|,0}$, $w_{|M|+1,n-1-|M|,0}$, and $w_{|M|,n-1-|M|,1}$ typically do not differ much, the normalized measures of decisiveness usually deliver values very close to each other. This can also by verified by inspecting the tables in the appendix.

5 Special state spaces: from zero to extreme correlation

As we have seen above, the votes will be uncorrelated if and only if the state space is perfectly balanced. In this case, all probability-based measures do not depend on the particular perfectly balanced state space, only the distance-based do so. The case of no correlation is of course the set-up underlying the development of the Banzhaf index of power which we found to agree in this situation with all normalized measures of decisiveness, δ, σ, and Ψ.

Another interesting special case is the state space $X = \mathbb{R}$ with Cauchy distributed random variables for status quo, proposal, and ideal points: as already stated earlier, in this case, the random variable $\pi(Q, P)$ describing the probability of a random voter preferring P over Q is uniformly distributed on $[0, 1]$. As is shown for instance in (Leech 1990), this is exactly the setting in which the Shapley-Shubik index of power

[21] We will discuss this issue in some detail and generality in the next subsection.

delivers the players' probabilities of being decisive: as can be seen from Tables 20 - 22,[22] we have in this case $\delta' = \delta = \phi$, with the Shapley-Shubik index of power being defined as

$$\phi_i := \sum_{M \in \widetilde{\mathrm{Piv}}_i} \frac{|M|!\,(n-1-|M|)!}{n!}$$

Notice, in particular, that all players' probabilities of being decisive sum up to unity which is not the case for other state spaces[23]. However, this does not mean that for every outcome of the vote, there is always exactly one player which is decisive. It only means that, on average, there is one player which is decisive: there are typically situations where several players are decisive, but also situations where no player is decisive.

We now consider an example that depends on a parameter in such a way that it is possible to study the effect of rising correlation. To that end, we define the state space $X = \{0, 1\}$ as the set containing only the two points 0 and 1. For the measure μ on X, we assume that $\mu(\{1\}) = \frac{1}{2}(1 + \sqrt{\rho})$, with a parameter ρ that can vary between 0 and 1. It is then easy to see that $\xi(0,1) = \xi(1,0) = \overline{\xi} = \rho$ such that the votes' correlation will also be equal to ρ so that we can interpret the parameter ρ as governing the correlation of this particular state space. Notice that, for this particular state space, the average distance $\overline{d}_i$ between a player's ideal point and the outcome of the vote will be unity if player i is not successful and zero if the player succeeds. Therefore, for this particular example, we have $\overline{d}_i = 1 - \sigma'_i$. We will now study how the measures of success, decisiveness, luck, and power answer to changes in correlation, ρ.

Proposition 1. *For the state space $X = \{0, 1\}$ with $\mu(\{1\}) = \frac{1}{2}(1 + \sqrt{\rho})$ for $\rho \in [0, 1]$, we have:*

1. $w_{j,k,0} = \frac{(1-\rho)^{\min(j,k)}}{2^{j+k+1}} \left(\left(1 - \sqrt{\rho}\right)^{|j-k|} + \left(1 + \sqrt{\rho}\right)^{|j-k|} \right)$ *for all j, k, l,*
2. *if $\min(j, k) > 0$, then $w_{j,k,0}$ converges to zero for ρ approaching unity*
3. *for $j, k \neq 0$, $w_{j,0,0}$ and $w_{0,k,0}$ converge to one half for ρ approaching unity*
4. *for ρ approaching unity,*
   (a) $\sigma'_i \approx 1$, $\overline{d}_i \approx 0$ *for all $i \in N$,*

   (b) $$\delta'_i \approx \begin{cases} 0 & : \varnothing \in \mathrm{Ins}_i \wedge N \setminus \{i\} \in \mathrm{Disp}_i \\ \frac{1}{2} & : \varnothing \in \mathrm{Ins}_i \wedge N \setminus \{i\} \in \mathrm{Piv}_i \\ \frac{1}{2} & : \varnothing \in \mathrm{Piv}_i \wedge N \setminus \{i\} \in \mathrm{Disp}_i \\ 1 & : \varnothing \in \mathrm{Piv}_i \wedge N \setminus \{i\} \in \mathrm{Piv}_i \end{cases}$$

   (c) $$\lambda_i \approx \begin{cases} 1 & : \varnothing \in \mathrm{Ins}_i \wedge N \setminus \{i\} \in \mathrm{Disp}_i \\ \frac{1}{2} & : \varnothing \in \mathrm{Ins}_i \wedge N \setminus \{i\} \in \mathrm{Piv}_i \\ \frac{1}{2} & : \varnothing \in \mathrm{Piv}_i \wedge N \setminus \{i\} \in \mathrm{Disp}_i \\ 0 & : \varnothing \in \mathrm{Piv}_i \wedge N \setminus \{i\} \in \mathrm{Piv}_i \end{cases}$$

[22] Notice that for Cauchy distributed random variables, the first moment, i.e. the expected value, does not exist. Therefore all distance-based measures are not defined in this case.

[23] Inspecting the tables given in the appendix shows that the sum of the players' probabilities of being decisive may fall below or exceed unity, depending on the state space and the voting rules.

(d) *with the notations*

$$\underline{m} := \min\left(\left\{|M| : M \in \bigcup_{i=1}^{n} \widetilde{\mathrm{Piv}}_i\right\} \cup \left\{n-1-|M| : M \in \bigcup_{i=1}^{n} \widetilde{\mathrm{Piv}}_i\right\}\right)$$

and $\alpha_i := |\{M \in \widetilde{\mathrm{Piv}}_i : |M| = \underline{m} \vee |M| = n-1-\underline{m}\}|$, *we have:*

$$\sigma_i, \delta_i, \Psi_i \approx \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j} =: \zeta_i.$$

Instead of a formal proof, we will discuss how to obtain and use the results of the preceding proposition. As for the first result, it is an immediate consequence of the corresponding formula given in Corollary 4, using that $\xi(.,.) \equiv \rho$. The second and third assertion are easy consequences of the first, showing that, asymptotically for extreme correlation, the probability of votes being split converges to zero while the probability of unanimous 'Yes.' votes and that of unanimous 'No.' votes each will approach one half. This in particular implies that all players including the dummy player will asymptotically always be successful as almost always a unanimous voting result will occur: everyone gets exactly what he wants, simply because all voters want the same. Correspondingly, due to $\overline{d} = 1 - \sigma'$, the average distance between ideal points and outcome converges to zero.

With respect to decisiveness and luck, matters are a little bit more complicated: depending on $\varnothing \in \mathrm{Piv}_i$ or $\varnothing \in \mathrm{Ins}_i$ (player i alone can or cannot enforce the proposal by voting for it) and $N \setminus \{i\} \in \mathrm{Piv}_i$ or $N \setminus \{i\} \in \mathrm{Disp}_i$ (player i alone can or cannot stop the proposal by voting against it), player i's success in case of unanimous votes is either due to decisiveness or due to luck. All in all, we have to distinguish four different cases: first of all, for normal players, i.e. for players who can neither enforce the proposal ($\varnothing \in \mathrm{Ins}_i$ or, equivalently, $v(\{i\} = 0)$ nor the status quo ($N \setminus \{i\} \in \mathrm{Disp}_i$ or, equivalently, $v(N \setminus \{i\}) = 1$), decisiveness converges to zero while luck converges to unity. Again, this is plausible as asymptotically all other players cast the same vote and normal players are not decisive but lucky when all others vote unanimously. If a player has a right to veto ($v(N \setminus \{i\}) = 0$) but no positive omnipotence ($v(\{i\} = 0)$, then two scenarios happen asymptotically each with probability one half: either all players vote 'Yes.' in which case player i is decisive because of not using the right to veto, or all players vote 'No.' in which case player i is not decisive but lucky. Therefore, a veto player lacking positive omnipotence will in case of extreme correlation be decisive and lucky with probability 50% each. An analogous result holds for a player with positive omnipotence but lacking the right to veto against the proposal. Finally, for a dictator with both positive and negative omnipotence, decisiveness is always 100%, regardless of the votes' correlation. On the other hand, a dictator is never lucky.

Finally, let us explain and discuss the limit of the power indices given in the last assertion of the proposition: for all normalized measures of power, Corollary 4 shows that they are ratios of sums of $w_{j,k,l}$, with j, k, l depending on the measure at hand. The first assertion of the preceding proposition shows that these are of magnitude $(1-\rho)^{\min(j,k)}$ which is, for ρ close to unity, the smaller the larger $\min(j,k)$ is. Asymptotically, therefore, only those sets M will play a role for which the exponent of $1-\rho$ is minimal:

this minimal exponent is denoted by $\underline{m}$, it gives the minimum taken over the number of voters that do not agree with the majority of voters in situations where at least one player is decisive. The corresponding limit index which we call extreme correlation power index (ECPI) takes only those situations into account where the number of voters voting against the majority is equal to $\underline{m}$ as these situations are asymptotically the only relevant situations. One may therefore view the extreme correlation power index ζ as the antipode of the Banzhaf index: while the Banzhaf index treats all situations as equally important, the limit index ζ completely ignores many situations in which players are decisive, taking into account only those where the number of voters disagreeing with the majority's vote is minimal.

To study the effect of increasing correlation, we present in the appendix two sets of tables (Tables 23 - 40) for two different voting rules and different values of ρ, increasing from the case of zero correlation, $\rho = 0$, to very high correlation, $\rho \approx 0.979$: for the first set of tables, 6 players have voting weights $(2, 2, 21, 28, 31, 31)$ and need 87 votes, i.e. an acceptance rate of 75%, for the proposal to pass, while for the second set, the voting weights are $(1, 1, 2, 3, 3, 10)$ with 11 votes necessary for the proposal to be accepted. The first set (Tables 23-31) shows the effects that one would expect by combining the results of Corollary 3 and Proposition 1:

- success is an increasing function of correlation, starting at $\sigma_i' = \frac{1}{2} + \left(\frac{1}{2}\right)^{n+1} |\,\mathrm{Piv}_i\,|$ for zero correlation and increasing to $\sigma_i' \approx 1$ for maximal correlation,

- for all players, decisiveness decreases from $\delta_i' = \left(\frac{1}{2}\right)^{n} |\,\mathrm{Piv}_i\,|$ in the absence of correlation to zero for extreme positive correlation (all players are normal players, lacking the power to enforce or prohibit the proposal),

- for all players, luck increases from $\lambda_i = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1} |\,\mathrm{Piv}_i\,|$ in case of zero correlation to unity for correlation converging to unity,

- all normalized measures of power start at the Banzhaf value β for zero correlation, move into the direction of the Shapley-Shubik value ϕ when correlation increases moderately, and, for large correlation, pass the values of the Shapley-Shubik index and converge eventually to the limit index ζ.

However, as the second set of examples (Tables 32-40) shows, the behavior of the measures of success, decisiveness, and power may also be more complicated:

- for player 6, success first decreases with increasing correlation, from 98.44% in the absence of correlation to 96.65% when $\rho \approx 0.456$. Only then success increases with correlation to its limit value of 100%,

- for players 1-5, decisiveness first increases with correlation, from 3.12% in the absence of correlation to 4.17% for $\rho = \frac{1}{3}$, later on decreasing to its limit of zero,

- the normalized measures of power start by taking the Banzhaf value for $\rho = 0$ and then moving towards the Shapley-Shubik value which they take for $\rho = \frac{3}{7} - \frac{2}{35}\sqrt{30} \approx 0.116$, further rising correlation sees the weaker players 1-5 gaining and player 6 losing power until ρ reaches the value $\frac{11}{5} - \frac{2}{5}\sqrt{19} \approx 0.456$ after which point the powerful player 6 gains power at the expense of the weaker players 1-5. At

$\rho = \frac{3}{7} + \frac{2}{35}\sqrt{30} \approx 0.742$, the normalized measures again agree with the Shapley-Shubik index, and only for very large correlations exceeding this value do they finally approach their limit value determined by the limit index ζ.

We would like to emphasize that the preceding example in particular shows that the normalized measures may take the same value for different correlations as $\rho = \frac{3}{7} - \frac{2}{35}\sqrt{30} \approx 0.116$ and $\rho = \frac{3}{7} + \frac{2}{35}\sqrt{30} \approx 0.742$ both deliver the Shapley-Shubik values. On the other hand, the example also shows that the value of the votes' correlation alone does not contain enough information to determine the measures of success, decisiveness, and power: for both Table 22 and Table 35, the correlation of the votes is $\frac{1}{3}$, but with completely different values of all measures of success, decisiveness, luck, and power.

To finish this section, we present the general result for correlation converging to unity:[24]

Proposition 2. *For $\xi(Q,P)$ approaching unity, we have:*

1. $w_{j,k,l} \approx \dfrac{E\left((z(Q,P)^l(1-\xi(Q,P))^{\min(j,k)}\right)}{2^{2\min(j,k)+1_{j\neq k}}}$ *for all* j,k,l,
2. $\sigma'_i \approx 1$ *for all* $i \in N$,
3. $\delta'_i \approx \begin{cases} 0 & : \varnothing \in \mathrm{Ins}_i \wedge N\setminus\{i\} \in \mathrm{Disp}_i \\ \frac{1}{2} & : \varnothing \in \mathrm{Ins}_i \wedge N\setminus\{i\} \in \mathrm{Piv}_i \\ \frac{1}{2} & : \varnothing \in \mathrm{Piv}_i \wedge N\setminus\{i\} \in \mathrm{Disp}_i \\ 1 & : \varnothing \in \mathrm{Piv}_i \wedge N\setminus\{i\} \in \mathrm{Piv}_i \end{cases}$
4. $\lambda_i \approx \begin{cases} 1 & : \varnothing \in \mathrm{Ins}_i \wedge N\setminus\{i\} \in \mathrm{Disp}_i \\ \frac{1}{2} & : \varnothing \in \mathrm{Ins}_i \wedge N\setminus\{i\} \in \mathrm{Piv}_i \\ \frac{1}{2} & : \varnothing \in \mathrm{Piv}_i \wedge N\setminus\{i\} \in \mathrm{Disp}_i \\ 0 & : \varnothing \in \mathrm{Piv}_i \wedge N\setminus\{i\} \in \mathrm{Piv}_i \end{cases}$
5. $\overline{d}_i \approx E(\min(d(Q,X_i),d(P,X_i)))$
6. $\sigma_i, \delta_i, \Psi_i \approx \zeta_i$.

Essentially, Proposition 2 is just the generalization of Proposition 1 to the general case, its only new content is the result for the average distance between outcome and players' ideal points: corresponding to the asymptotic probability of 100% of being successful, players can profit from extreme correlation by bringing their average distance down to the expected distance of the minimum of the distances of their ideal point to the status quo and the proposal, respectively: they are almost always able to get what they wish for, i.e. the outcome that minimizes the distance to the ideal point. All in all, we have discussed two groups of power indices consisting of three members each: first, the correlation-insensitive indices, Banzhaf, Shapley-Shubik, and extreme correlation power

[24]Formally, the random variable $\xi(Q,P)$ has to converge to unity in a suitable sense. However, we will omit these technicalities and just state the result without a formal proof.

index (ECPI), and, second, the correlation-sensitive indizes, normalized decisiveness, normalized excess success, and normalized reduction of average distance (SPI), which take the state space's imbalance into account. The correlation-insensitive power indices correspond to specific situations: the Banzhaf corresponds to zero correlation, the ECPI is developed for extremely high correlation close to unity, and the Shapley-Shubik corresponds to a moderate correlation of $\frac{1}{3}$ produced by uniformly distributed $\Pi = \pi(Q, P)$. Within the second group of indices, values typically do not differ by much, so choosing a particular one is mostly a matter of taste. For the case of zero correlation, they all agree with the Banzhaf, for extremely high correlation they will all be close to the ECPI. For moderate correlation, they might be approximated by the Shapley-Shubik index, but as Tables 35-38 show, there is no guarantee that this approximation is actually adequate.

6 Comparative statics with respect to the voting rules

In the absence of correlation, due to all measures of success and decisiveness depending on $|\,\mathrm{Piv}_i\,|$ only, player i's only way of improving her situation is given by increasing the set Piv_i, i.e. by a change of rules such that there are more constellations where player i is decisive. When such a change is set into force, all measures of decisiveness and success will simultaneously improve for player i.

In the presence of positive correlation, however, success and decisiveness need no longer go hand in hand: for instance, for the two voting rules underlying Tables 7 and 10, we find that players 5 and 6 are much less decisive in the situation of Table 10, but they are more often successful than in the situation of Table 7. In some way, therefore, by moving from the voting role corresponding to Table 7 to that of Table 10, players 5 and 6 have sacrificed decisiveness for the sake of gaining success. To study the reasons underlying this phenomenon, we first state a quite technical proposition whose interesting implications we will discuss below.

Proposition 3.

1. *for all* j, k, l, *we have:* $w_{j,k,l} = w_{k,j,l}$
2. *for all* $m, \tilde{n}, l$ *with* $m + 1 \leq \tilde{n}$, *we have:*
   (a) $w_{m+1,\tilde{n}-m,l} + w_{m,\tilde{n}+1-m,l} = w_{m,\tilde{n}-m,l}$,
   (b) $w_{m,\tilde{n}-m,l} \geq w_{m+1,\tilde{n}-(m+1),l}$ *if and only if* $m \leq \frac{\tilde{n}-1}{2}$
3. *for the dual game* v^* *defined by* $v^*(M) := 1 - v(N \setminus M)$ *and the corresponding sets* Ins_i^*, Disp_i^*, *and* Piv_i^* *of situations where player i is insufficient, dispensable, and decisive, resp., with respect to the dual game, we have:*
   (a) $M \in \mathrm{Ins}_i^*$ *if and only if* $N \setminus (M \cup \{i\}) \in \mathrm{Disp}_i$,
   (b) $M \in \mathrm{Disp}_i^*$ *if and only if* $N \setminus (M \cup \{i\}) \in \mathrm{Ins}_i$,
   (c) $M \in \mathrm{Piv}_i^*$ *if and only if* $N \setminus (M \cup \{i\}) \in \mathrm{Piv}_i$,

4. *when v is replaced with v^*, all measures of luck, decisiveness, success, and power stay the same.*

Proof. See appendix B.3. □

Using 2.(a) of the proposition, we might rewrite the formula for success[25],

$$\sigma_i' = \sum_{M \in \mathrm{Disp}_i} w_{|M|+1,n-|M|,0} + \sum_{M \in \mathrm{Ins}_i} w_{|M|,n+1-|M|,0} + \sum_{M \in \mathrm{Piv}_i} w_{|M|,n-|M|,0}$$

in the following way:

$$\begin{aligned}
\sigma_i' &= \sum_{M \subseteq N \setminus \{i\}} \left(w_{|M|+1,n-|M|,0} 1_{M \in \mathrm{Disp}_i} + w_{|M|,n+1-|M|,0} 1_{M \in \mathrm{Ins}_i} + w_{|M|,n-|M|,0} 1_{M \in \mathrm{Piv}_i} \right) \\
&= \sum_{M \subseteq N \setminus \{i\}} \left(w_{|M|+1,n-|M|,0} 1_{M \in \mathrm{Disp}_i \cup \mathrm{Piv}_i} + w_{|M|,n+1-|M|,0} 1_{M \in \mathrm{Ins}_i \cup \mathrm{Piv}_i} \right)
\end{aligned}$$

This formula is highly useful, it can be exploited in several ways: for instance, in terms of the contribution to player i's success, we find using the proposition that

- for fixed M, it is best if $M \in \mathrm{Piv}_i$ as the contribution of M then is the sum of those where $M \in \mathrm{Ins}_i$ and $M \in \mathrm{Disp}_i$,
- for fixed M, $M \in \mathrm{Ins}_i$ delivers a larger contribution than $M \in \mathrm{Disp}_i$ if and only if $|M| \leq \frac{n}{2}$,
- if $M \in \mathrm{Piv}_i$, the size of M is important: the closer $|M|$ to $\frac{n}{2}$, the smaller the contribution of M to success as well as decisiveness,
- if $M \in \mathrm{Disp}_i$, the size of M is important: the closer $|M|$ to $\frac{n-1}{2}$, the smaller the contribution of M to success,
- if $M \in \mathrm{Ins}_i$, the size of M is important: the closer $|M|$ to $\frac{n+1}{2}$, the smaller the contribution of M to success.

One consequence of the above is the following: for the never decisive dummy player ($\mathrm{Piv}_0 = \varnothing$), it is best if all small M, i.e. those with $|M| \leq \frac{n}{2}$, belong to Ins_0, i.e. lead to the proposal being dismissed, and all large M, i.e. those with $|M| \geq \frac{n}{2}$, belong to Disp_0, i.e. lead to the proposal being accepted. As $|M|$ is the number of votes in favor of the proposal, this rule maximizing the dummy's success is of course nothing else than the rule of simple majority per capita. For the example of Table 10, the dummy player thereby achieves a success rate of more than 62%, nota bene without any actual influence on the outcome of the vote. Phrased differently, the voting rules in combination with the positive correlation lead to the actual voters producing some kind of positive externality to the benefit of powerless voters.

Another consequence of the above is that it is more important to be decisive in situations with either very few or a lot of supporters of the proposal, while it is less important to be decisive in situations where the voters are split into two almost equally

[25] We restrict this discussion to the measures σ' and δ'. Similar assertions hold for the distance-based measures.

large groups of supporters and opponents of the proposal. This is also completely in line with intuition: the latter case of two equally large groups is less likely to occur due to the positive correlation of the votes. Note that this fact may have consequences completely different from the case of zero correlation where it is only important to be decisive in as many situations as possible: it may now happen that under a new voting rule, the number $|\,\mathrm{Piv}_i\,|$ of situations with player i being decisive increases, although at the same time the probability of the player being decisive decreases: we leave it to the reader to verify that by changing the voting rule of Table 7 to that of Table 10, $|\,\mathrm{Piv}_5\,| = |\,\mathrm{Piv}_6\,|$ increases from 18 to 20, although players 5 and 6's probability of being decisive decreases by almost 10%.

Let us now suppose that the players engage in negotiations about a new constitution concerning the voting rules. Then it is very well possible that there is room for all players to improve their probability of being successful by cleverly combining the ideas discussed above: some players may profit from being decisive in more situations, some may trade $M \in \mathrm{Piv}_i$ with $|M| \approx \frac{n}{2}$ for another $M \in \mathrm{Piv}_i$ with $|M|$ either small or large, and others may benefit from the new rules because some of the situations $M \subseteq N \setminus \{i\}$ move from Ins_i to Disp_i or vice versa. Again, Tables 7 and 10 provide such an example: under the simple majority per capita, all players have a higher probability of being successful than under the voting rule of Table 7, although players 5 and 6 are now much less decisive: this is compensated by these players now being more lucky than before.

Similarly, as the tables in the appendix reveal, all players are better of in terms of success if the qualified majority rule requiring 75% of the weights for a proposal ($\gamma = 86.25$) is replaced by the less stringent rule necessitating only 60 votes ($\gamma = 59.5$). From the point of maximizing the probability of being successful, the latter voting rule therefore is Pareto superior to the former.

7 Conclusion

In this paper, we analyze a source of correlation between the votes of independent voters that has not been discussed in the literature so far: the geometry of the state space. We develop a new concept for the description of a state space, labeled perfect balancedness, and show that perfect balancedness of a state space is a necessary and sufficient condition for non-correlation of the votes of independent voters. The votes of independent voters are necessarily positively correlated if the state space is not perfectly balanced. Our analysis also sheds a new light on Barry's formula of 'success = luck + power', in terms of probabilities, distances, and utilities. We show that one can transform a measure of success into one for decisiveness by relating it to a dummy player's value for success. Moreover, we develop formulas for calculating all these quantities for arbitrary state spaces, showing that the distribution of imbalance across a state space is decisive for these measures.

By normalizing decisiveness, excess success, and the reduction of average distances, we consider three power measures whose values are usually quite close to each other: normalized excess success is an entirely new measure while normalized reduction of average distances is known as the Strategic Power Index. When correlation converges to unity, all these measures approach the new extreme correlation power index (ECPI) which is an antipode to the Banzhaf. The Banzhaf, Shapley-Shubik, and ECPI index are all correlation-insensitive measures of power which in particular situations might be good approximations to the three correlation-sensitive power indices. Furthermore, all correlation-sensitive indices agree with the Banzhaf for zero correlation, therefore they may be seen as generalizations of the Banzhaf index.

Finally we provide some comparative statics results w.r.t. to voting rules, showing that for positive correlation, the number of situations where some player is decisive is not enough to determine this player's probability of being decisive: situations with the voting body split into two almost equally large groups are less important, those with almost unanimous votes are more important. Moreover, powerless players are best off when the other players decide using the simple majority rule.

Quite a number of questions arise immediately:

- The analysis of this paper is based on the simplifying assumption of identical probability distributions for status quo, proposals, and ideal points. What will be the consequences if these probability distributions are different?

- A very important question is how the different sources of correlation of votes known from the literature interact with the correlation due to the geometry of the state space. How would our results change if we take into account other sources of correlation?

- We did not distinguish between winning (positive) power and rejecting (negative) power. It would be interesting to define indices for winning and rejecting power based on the notion of excess success and to show how they react to the geometry of the state space.

- We analyzed a very simple game: the voting game has been one-shot, proposals to be voted upon 'fell from heaven', there has been no agenda setter. Moreover, it has been a dominant strategy for all voters to vote for that alternative which is preferred according to the distance to a voter's ideal point. Relaxing these assumptions and analyzing more complex games with the possibility of strategic voting would enable us to deliver a much more complete picture of state space generated correlations of votes and voting power. This will have important ramifications for the choice of voting rules.

We leave these interesting issues for future research.

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A Additional Tables

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	50.00	46.88	46.88	46.88	28.12	25.00	25.00
succ. σ'	50.00	53.12	53.12	53.12	71.88	75.00	75.00
avg. dist. $\overline{d}$	1.273	1.252	1.252	1.252	1.121	1.099	1.099
dec. δ'	0	6.25	6.25	6.25	43.75	50.00	50.00
exc. succ. σ^E	0	3.12	3.12	3.12	21.88	25.00	25.00
red. avg. dist. Ψ'	0	0.022	0.022	0.022	0.152	0.174	0.174
norm. dec. δ	0	3.85	3.85	3.85	26.92	30.77	30.77
norm. exc. succ. σ	0	3.85	3.85	3.85	26.92	30.77	30.77
SPI Ψ	0	3.85	3.85	3.85	26.92	30.77	30.77
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 2: Circle, Line, Corr=0, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	58.59	55.92	55.92	55.92	38.60	35.28	35.28
succ. σ'	58.59	60.89	60.89	60.89	75.41	78.09	78.09
avg. dist. $\overline{d}$	0.863	0.852	0.852	0.852	0.784	0.772	0.772
dec. δ'	0	4.97	4.97	4.97	36.82	42.81	42.81
exc. succ. σ^E	0	2.30	2.30	2.30	16.82	19.49	19.49
red. avg. dist. Ψ'	0	0.011	0.011	0.011	0.079	0.091	0.091
norm. dec. δ	0	3.62	3.62	3.62	26.81	31.17	31.17
norm. exc. succ. σ	0	3.66	3.66	3.66	26.83	31.09	31.09
SPI Ψ	0	3.70	3.70	3.70	26.85	31.03	31.03
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 3: Circle, Area, Corr=0.123, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	61.26	58.76	58.76	58.76	42.08	38.75	38.75
succ. σ'	61.26	63.34	63.34	63.34	76.66	79.17	79.17
avg. dist. $\overline{d}$	0.310	0.304	0.304	0.304	0.267	0.261	0.261
dec. δ'	0	4.58	4.58	4.58	34.58	40.42	40.42
exc. succ. σ^E	0	2.08	2.08	2.08	15.39	17.90	17.90
red. avg. dist. Ψ'	0	0.006	0.006	0.006	0.042	0.049	0.049
norm. dec. δ	0	3.55	3.55	3.55	26.77	31.29	31.29
norm. exc. succ. σ	0	3.61	3.61	3.61	26.81	31.17	31.17
SPI Ψ	0	3.68	3.68	3.68	26.84	31.07	31.07
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 4: Line, Corr=$\frac{1}{6}$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	50.00	48.44	48.44	48.44	39.06	35.94	35.94
succ. σ'	50.00	51.56	51.56	51.56	60.94	64.06	64.06
avg. dist. $\overline{d}$	1.273	1.262	1.262	1.262	1.197	1.175	1.175
dec. δ'	0	3.12	3.12	3.12	21.88	28.12	28.12
exc. succ. σ^E	0	1.56	1.56	1.56	10.94	14.06	14.06
red. avg. dist. Ψ'	0	0.011	0.011	0.011	0.076	0.098	0.098
norm. dec. δ	0	3.57	3.57	3.57	25.00	32.14	32.14
norm. exc. succ. σ	0	3.57	3.57	3.57	25.00	32.14	32.14
SPI Ψ	0	3.57	3.57	3.57	25.00	32.14	32.14
Banzhaf β	0	3.57	3.57	3.57	25.00	32.14	32.14
Shapley-Shubik ϕ	0	3.33	3.33	3.33	16.67	36.67	36.67
ECPI ζ	0	0.00	0.00	0.00	0.00	50.00	50.00

Table 5: Circle, Line, Corr=0, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 86.25$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	55.80	53.81	53.81	53.81	44.51	37.71	37.71
succ. σ'	55.80	57.32	57.32	57.32	64.96	69.30	69.30
avg. dist. $\overline{d}$	0.878	0.871	0.871	0.871	0.836	0.818	0.818
dec. δ'	0	3.51	3.51	3.51	20.45	31.59	31.59
exc. succ. σ^E	0	1.52	1.52	1.52	9.16	13.50	13.50
red. avg. dist. Ψ'	0	0.007	0.007	0.007	0.042	0.060	0.060
norm. dec. δ	0	3.72	3.72	3.72	21.72	33.55	33.55
norm. exc. succ. σ	0	3.74	3.74	3.74	22.49	33.15	33.15
SPI Ψ	0	3.73	3.73	3.73	23.00	32.91	32.91
Banzhaf β	0	3.57	3.57	3.57	25.00	32.14	32.14
Shapley-Shubik ϕ	0	3.33	3.33	3.33	16.67	36.67	36.67
ECPI ζ	0	0.00	0.00	0.00	0.00	50.00	50.00

Table 6: Circle, Area, Corr=0.123, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 86.25$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	57.90	55.83	55.83	55.83	46.67	38.33	38.33
succ. σ'	57.90	59.37	59.37	59.37	66.46	71.04	71.04
avg. dist. $\overline{d}$	0.318	0.314	0.314	0.314	0.295	0.285	0.285
dec. δ'	0	3.54	3.54	3.54	19.79	32.71	32.71
exc. succ. σ^E	0	1.47	1.47	1.47	8.56	13.14	13.14
red. avg. dist. Ψ'	0	0.004	0.004	0.004	0.023	0.033	0.033
norm. dec. δ	0	3.70	3.70	3.70	20.65	34.13	34.13
norm. exc. succ. σ	0	3.74	3.74	3.74	21.81	33.48	33.48
SPI Ψ	0	3.74	3.74	3.74	22.72	33.03	33.03
Banzhaf β	0	3.57	3.57	3.57	25.00	32.14	32.14
Shapley-Shubik ϕ	0	3.33	3.33	3.33	16.67	36.67	36.67
ECPI ζ	0	0.00	0.00	0.00	0.00	50.00	50.00

Table 7: Line, Corr=$\frac{1}{6}$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 86.25$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	50.00	34.38	34.38	34.38	34.38	34.38	34.38
succ. σ'	50.00	65.62	65.62	65.62	65.62	65.62	65.62
avg. dist. $\overline{d}$	1.273	1.165	1.165	1.165	1.165	1.165	1.165
dec. δ'	0	31.25	31.25	31.25	31.25	31.25	31.25
exc. succ. σ^E	0	15.62	15.62	15.62	15.62	15.62	15.62
red. avg. dist. Ψ'	0	0.109	0.109	0.109	0.109	0.109	0.109
norm. dec. δ	0	16.67	16.67	16.67	16.67	16.67	16.67
norm. exc. succ. σ	0	16.67	16.67	16.67	16.67	16.67	16.67
SPI Ψ	0	16.67	16.67	16.67	16.67	16.67	16.67
Banzhaf β	0	16.67	16.67	16.67	16.67	16.67	16.67
Shapley-Shubik ϕ	0	16.67	16.67	16.67	16.67	16.67	16.67
ECPI ζ	0	16.67	16.67	16.67	16.67	16.67	16.67

Table 8: Circle, Line, Corr=0, $\mathbf{w} = (1,1,1,1,1,1)$, $\gamma = 3.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	59.34	45.99	45.99	45.99	45.99	45.99	45.99
succ. σ'	59.34	70.82	70.82	70.82	70.82	70.82	70.82
avg. dist. $\overline{d}$	0.859	0.805	0.805	0.805	0.805	0.805	0.805
dec. δ'	0	24.84	24.84	24.84	24.84	24.84	24.84
exc. succ. σ^E	0	11.48	11.48	11.48	11.48	11.48	11.48
red. avg. dist. Ψ'	0	0.054	0.054	0.054	0.054	0.054	0.054
norm. dec. δ	0	16.67	16.67	16.67	16.67	16.67	16.67
norm. exc. succ. σ	0	16.67	16.67	16.67	16.67	16.67	16.67
SPI Ψ	0	16.67	16.67	16.67	16.67	16.67	16.67
Banzhaf β	0	16.67	16.67	16.67	16.67	16.67	16.67
Shapley-Shubik ϕ	0	16.67	16.67	16.67	16.67	16.67	16.67
ECPI ζ	0	16.67	16.67	16.67	16.67	16.67	16.67

Table 9: Circle, Area, Corr=0.123, $\mathbf{w} = (1,1,1,1,1,1)$, $\gamma = 3.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	62.13	49.59	49.59	49.59	49.59	49.59	49.59
succ. σ'	62.13	72.51	72.51	72.51	72.51	72.51	72.51
avg. dist. $\overline{d}$	0.307	0.278	0.278	0.278	0.278	0.278	0.278
dec. δ'	0	22.92	22.92	22.92	22.92	22.92	22.92
exc. succ. σ^E	0	10.38	10.38	10.38	10.38	10.38	10.38
red. avg. dist. Ψ'	0	0.029	0.029	0.029	0.029	0.029	0.029
norm. dec. δ	0	16.67	16.67	16.67	16.67	16.67	16.67
norm. exc. succ. σ	0	16.67	16.67	16.67	16.67	16.67	16.67
SPI Ψ	0	16.67	16.67	16.67	16.67	16.67	16.67
Banzhaf β	0	16.67	16.67	16.67	16.67	16.67	16.67
Shapley-Shubik ϕ	0	16.67	16.67	16.67	16.67	16.67	16.67
ECPI ζ	0	16.67	16.67	16.67	16.67	16.67	16.67

Table 10: Line, Corr=$\frac{1}{6}$, $\mathbf{w} = (1,1,1,1,1,1)$, $\gamma = 3.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	50.00	46.88	46.88	46.88	28.12	25.00	25.00
succ. σ'	50.00	53.12	53.12	53.12	71.88	75.00	75.00
avg. dist. $\overline{d}$	1.309	1.269	1.269	1.269	1.028	0.988	0.988
dec. δ'	0	6.25	6.25	6.25	43.75	50.00	50.00
exc. succ. σ^E	0	3.12	3.12	3.12	21.88	25.00	25.00
red. avg. dist. Ψ'	0	0.040	0.040	0.040	0.281	0.321	0.321
norm. dec. δ	0	3.85	3.85	3.85	26.92	30.77	30.77
norm. exc. succ. σ	0	3.85	3.85	3.85	26.92	30.77	30.77
SPI Ψ	0	3.85	3.85	3.85	26.92	30.77	30.77
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 11: Rectangle, Vertices, Corr=0, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	53.83	50.88	50.88	50.88	32.56	29.30	29.30
succ. σ'	53.83	56.55	56.55	56.55	73.30	76.25	76.25
avg. dist. $\overline{d}$	1.065	1.048	1.048	1.048	0.942	0.924	0.924
dec. δ'	0	5.66	5.66	5.66	40.74	46.95	46.95
exc. succ. σ^E	0	2.71	2.71	2.71	19.47	22.42	22.42
red. avg. dist. Ψ'	0	0.017	0.017	0.017	0.123	0.141	0.141
norm. dec. δ	0	3.74	3.74	3.74	26.87	30.96	30.96
norm. exc. succ. σ	0	3.74	3.74	3.74	26.87	30.95	30.95
SPI Ψ	0	3.77	3.77	3.77	26.89	30.90	30.90
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 12: Rectangle, Edges, Corr=0.05, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	60.13	57.55	57.55	57.55	40.56	37.23	37.23
succ. σ'	60.13	62.29	62.29	62.29	76.11	78.69	78.69
avg. dist. $\overline{d}$	0.758	0.748	0.748	0.748	0.682	0.670	0.670
dec. δ'	0	4.74	4.74	4.74	35.54	41.46	41.46
exc. succ. σ^E	0	2.16	2.16	2.16	15.98	18.56	18.56
red. avg. dist. Ψ'	0	0.010	0.010	0.010	0.076	0.088	0.088
norm. dec. δ	0	3.58	3.58	3.58	26.79	31.24	31.24
norm. exc. succ. σ	0	3.63	3.63	3.63	26.82	31.15	31.15
SPI Ψ	0	3.68	3.68	3.68	26.84	31.07	31.07
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 13: Rectangle, Area, Corr=0.148, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	50.00	48.44	48.44	48.44	39.06	35.94	35.94
succ. σ'	50.00	51.56	51.56	51.56	60.94	64.06	64.06
avg. dist. $\overline{d}$	1.309	1.289	1.289	1.289	1.169	1.128	1.128
dec. δ'	0	3.12	3.12	3.12	21.88	28.12	28.12
exc. succ. σ^E	0	1.56	1.56	1.56	10.94	14.06	14.06
red. avg. dist. Ψ'	0	0.020	0.020	0.020	0.141	0.181	0.181
norm. dec. δ	0	3.57	3.57	3.57	25.00	32.14	32.14
norm. exc. succ. σ	0	3.57	3.57	3.57	25.00	32.14	32.14
SPI Ψ	0	3.57	3.57	3.57	25.00	32.14	32.14
Banzhaf β	0	3.57	3.57	3.57	25.00	32.14	32.14
Shapley-Shubik ϕ	0	3.33	3.33	3.33	16.67	36.67	36.67
ECPI ζ	0	0.00	0.00	0.00	0.00	50.00	50.00

Table 14: Rectangle, Vertices, Corr=0, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 86.25$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	52.27	50.49	50.49	50.49	41.02	36.65	36.65
succ. σ'	52.27	53.86	53.86	53.86	62.48	66.28	66.28
avg. dist. $\overline{d}$	1.075	1.065	1.065	1.065	1.011	0.989	0.989
dec. δ'	0	3.38	3.38	3.38	21.46	29.63	29.63
exc. succ. σ^E	0	1.60	1.60	1.60	10.21	14.01	14.01
red. avg. dist. Ψ'	0	0.010	0.010	0.010	0.064	0.086	0.086
norm. dec. δ	0	3.72	3.72	3.72	23.62	32.61	32.61
norm. exc. succ. σ	0	3.71	3.71	3.71	23.73	32.57	32.57
SPI Ψ	0	3.68	3.68	3.68	24.08	32.44	32.44
Banzhaf β	0	3.57	3.57	3.57	25.00	32.14	32.14
Shapley-Shubik ϕ	0	3.33	3.33	3.33	16.67	36.67	36.67
ECPI ζ	0	0.00	0.00	0.00	0.00	50.00	50.00

Table 15: Rectangle, Edges, Corr=0.05, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 86.25$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	56.96	54.92	54.92	54.92	45.67	38.05	38.05
succ. σ'	56.96	58.46	58.46	58.46	65.78	70.28	70.28
avg. dist. $\overline{d}$	0.774	0.767	0.767	0.767	0.733	0.715	0.715
dec. δ'	0	3.54	3.54	3.54	20.11	32.23	32.23
exc. succ. σ^E	0	1.50	1.50	1.50	8.82	13.32	13.32
red. avg. dist. Ψ'	0	0.007	0.007	0.007	0.041	0.060	0.060
norm. dec. δ	0	3.72	3.72	3.72	21.13	33.86	33.86
norm. exc. succ. σ	0	3.75	3.75	3.75	22.08	33.34	33.34
SPI Ψ	0	3.74	3.74	3.74	22.73	33.03	33.03
Banzhaf β	0	3.57	3.57	3.57	25.00	32.14	32.14
Shapley-Shubik ϕ	0	3.33	3.33	3.33	16.67	36.67	36.67
ECPI ζ	0	0.00	0.00	0.00	0.00	50.00	50.00

Table 16: Rectangle, Area, Corr=0.148, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 86.25$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	50.00	48.44	48.44	48.44	48.44	48.44	1.56
succ. σ'	50.00	51.56	51.56	51.56	51.56	51.56	98.44
avg. dist. $\overline{d}$	1.309	1.289	1.289	1.289	1.289	1.289	0.687
dec. δ'	0	3.12	3.12	3.12	3.12	3.12	96.88
exc. succ. σ^E	0	1.56	1.56	1.56	1.56	1.56	48.44
red. avg. dist. Ψ'	0	0.020	0.020	0.020	0.020	0.020	0.622
norm. dec. δ	0	2.78	2.78	2.78	2.78	2.78	86.11
norm. exc. succ. σ	0	2.78	2.78	2.78	2.78	2.78	86.11
SPI Ψ	0	2.78	2.78	2.78	2.78	2.78	86.11
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 17: Rectangle, Vertices, Corr=0, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	52.81	51.03	51.03	51.03	51.03	51.03	2.89
succ. σ'	52.81	54.41	54.41	54.41	54.41	54.41	98.10
avg. dist. $\overline{d}$	1.071	1.062	1.062	1.062	1.062	1.062	0.789
dec. δ'	0	3.38	3.38	3.38	3.38	3.38	95.21
exc. succ. σ^E	0	1.60	1.60	1.60	1.60	1.60	45.29
red. avg. dist. Ψ'	0	0.010	0.010	0.010	0.010	0.010	0.282
norm. dec. δ	0	3.01	3.01	3.01	3.01	3.01	84.94
norm. exc. succ. σ	0	2.99	2.99	2.99	2.99	2.99	85.03
SPI Ψ	0	2.94	2.94	2.94	2.94	2.94	85.30
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 18: Rectangle, Edges, Corr=0.05, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	58.13	56.09	56.09	56.09	56.09	56.09	6.33
succ. σ'	58.13	59.63	59.63	59.63	59.63	59.63	97.75
avg. dist. $\overline{d}$	0.768	0.762	0.762	0.762	0.762	0.762	0.585
dec. δ'	0	3.54	3.54	3.54	3.54	3.54	91.42
exc. succ. σ^E	0	1.50	1.50	1.50	1.50	1.50	39.62
red. avg. dist. Ψ'	0	0.007	0.007	0.007	0.007	0.007	0.183
norm. dec. δ	0	3.24	3.24	3.24	3.24	3.24	83.78
norm. exc. succ. σ	0	3.18	3.18	3.18	3.18	3.18	84.10
SPI Ψ	0	3.11	3.11	3.11	3.11	3.11	84.44
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 19: Rectangle, Area, Corr=0.148, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	70.48	68.57	68.57	68.57	54.76	51.67	51.67
succ. σ'	70.48	71.90	71.90	71.90	81.43	83.33	83.33
avg. dist. $\overline{d}$							
dec. δ'	0	3.33	3.33	3.33	26.67	31.67	31.67
exc. succ. σ^E	0	1.43	1.43	1.43	10.95	12.86	12.86
red. avg. dist. Ψ'							
norm. dec. δ	0	3.33	3.33	3.33	26.67	31.67	31.67
norm. exc. succ. σ	0	3.49	3.49	3.49	26.74	31.40	31.40
SPI Ψ							
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 20: Cauchy, Corr=$\frac{1}{3}$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	66.19	64.05	64.05	64.05	55.95	40.71	40.71
succ. σ'	66.19	67.38	67.38	67.38	72.62	77.38	77.38
avg. dist. $\overline{d}$							
dec. δ'	0	3.33	3.33	3.33	16.67	36.67	36.67
exc. succ. σ^E	0	1.19	1.19	1.19	6.43	11.19	11.19
red. avg. dist. Ψ'							
norm. dec. δ	0	3.33	3.33	3.33	16.67	36.67	36.67
norm. exc. succ. σ	0	3.68	3.68	3.68	19.85	34.56	34.56
SPI Ψ							
Banzhaf β	0	3.57	3.57	3.57	25.00	32.14	32.14
Shapley-Shubik ϕ	0	3.33	3.33	3.33	16.67	36.67	36.67
ECPI ζ	0	0.00	0.00	0.00	0.00	50.00	50.00

Table 21: Cauchy, Corr=$\frac{1}{3}$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 86.25$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	67.86	65.71	65.71	65.71	65.71	65.71	14.29
succ. σ'	67.86	69.05	69.05	69.05	69.05	69.05	97.62
avg. dist. $\overline{d}$							
dec. δ'	0	3.33	3.33	3.33	3.33	3.33	83.33
exc. succ. σ^E	0	1.19	1.19	1.19	1.19	1.19	29.76
red. avg. dist. Ψ'							
norm. dec. δ	0	3.33	3.33	3.33	3.33	3.33	83.33
norm. exc. succ. σ	0	3.33	3.33	3.33	3.33	3.33	83.33
SPI Ψ							
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 22: Cauchy, Corr=$\frac{1}{3}$, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	50.00	46.88	46.88	46.88	28.12	25.00	25.00
succ. σ'	50.00	53.12	53.12	53.12	71.88	75.00	75.00
avg. dist. $\overline{d}$	0.500	0.469	0.469	0.469	0.281	0.250	0.250
dec. δ'	0	6.25	6.25	6.25	43.75	50.00	50.00
exc. succ. σ^E	0	3.12	3.12	3.12	21.88	25.00	25.00
red. avg. dist. Ψ'	0	0.031	0.031	0.031	0.219	0.250	0.250
norm. dec. δ	0	3.85	3.85	3.85	26.92	30.77	30.77
norm. exc. succ. σ	0	3.85	3.85	3.85	26.92	30.77	30.77
SPI Ψ	0	3.85	3.85	3.85	26.92	30.77	30.77
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 23: Two Points, Corr=0, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	52.81	49.80	49.80	49.80	31.29	28.06	28.06
succ. σ'	52.81	55.62	55.62	55.62	72.87	75.88	75.88
avg. dist. $\overline{d}$	0.472	0.444	0.444	0.444	0.271	0.241	0.241
dec. δ'	0	5.82	5.82	5.82	41.57	47.82	47.82
exc. succ. σ^E	0	2.81	2.81	2.81	20.06	23.07	23.07
red. avg. dist. Ψ'	0	0.028	0.028	0.028	0.201	0.231	0.231
norm. dec. δ	0	3.76	3.76	3.76	26.88	30.92	30.92
norm. exc. succ. σ	0	3.76	3.76	3.76	26.88	30.92	30.92
SPI Ψ	0	3.76	3.76	3.76	26.88	30.92	30.92
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 24: Two Points, Corr=$\frac{1}{3} - \frac{2}{15}\sqrt{5} \approx 0.035$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	58.90	56.17	56.17	56.17	38.39	34.95	34.95
succ. σ'	58.90	61.06	61.06	61.06	75.16	77.89	77.89
avg. dist. $\overline{d}$	0.411	0.389	0.389	0.389	0.248	0.221	0.221
dec. δ'	0	4.89	4.89	4.89	36.78	42.94	42.94
exc. succ. σ^E	0	2.16	2.16	2.16	16.26	18.99	18.99
red. avg. dist. Ψ'	0	0.022	0.022	0.022	0.163	0.190	0.190
norm. dec. δ	0	3.56	3.56	3.56	26.78	31.27	31.27
norm. exc. succ. σ	0	3.56	3.56	3.56	26.78	31.27	31.27
SPI Ψ	0	3.56	3.56	3.56	26.78	31.27	31.27
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 25: Two Points, Corr=$\frac{3}{7} - \frac{2}{35}\sqrt{30} \approx 0.116$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	73.15	71.30	71.30	71.30	56.48	52.78	52.78
succ. σ'	73.15	74.07	74.07	74.07	81.48	83.33	83.33
avg. dist. $\overline{d}$	0.269	0.259	0.259	0.259	0.185	0.167	0.167
dec. δ'	0	2.78	2.78	2.78	25.00	30.56	30.56
exc. succ. σ^E	0	0.93	0.93	0.93	8.33	10.19	10.19
red. avg. dist. Ψ'	0	0.009	0.009	0.009	0.083	0.102	0.102
norm. dec. δ	0	2.94	2.94	2.94	26.47	32.35	32.35
norm. exc. succ. σ	0	2.94	2.94	2.94	26.47	32.35	32.35
SPI Ψ	0	2.94	2.94	2.94	26.47	32.35	32.35
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 26: Two Points, Corr=$\frac{1}{3}$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	75.04	73.32	73.32	73.32	59.07	55.37	55.37
succ. σ'	75.04	75.83	75.83	75.83	82.44	84.16	84.16
avg. dist. $\overline{d}$	0.250	0.242	0.242	0.242	0.176	0.158	0.158
dec. δ'	0	2.51	2.51	2.51	23.37	28.79	28.79
exc. succ. σ^E	0	0.80	0.80	0.80	7.41	9.12	9.12
red. avg. dist. Ψ'	0	0.008	0.008	0.008	0.074	0.091	0.091
norm. dec. δ	0	2.84	2.84	2.84	26.42	32.54	32.54
norm. exc. succ. σ	0	2.84	2.84	2.84	26.42	32.54	32.54
SPI Ψ	0	2.84	2.84	2.84	26.42	32.54	32.54
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 27: Two Points, Corr=$\frac{2}{15}\sqrt{105} - 1 \approx 0.366$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	79.87	78.52	78.52	78.52	65.94	62.33	62.33
succ. σ'	79.87	80.37	80.37	80.37	85.07	86.41	86.41
avg. dist. $\overline{d}$	0.201	0.196	0.196	0.196	0.149	0.136	0.136
dec. δ'	0	1.85	1.85	1.85	19.13	24.08	24.08
exc. succ. σ^E	0	0.50	0.50	0.50	5.20	6.54	6.54
red. avg. dist. Ψ'	0	0.005	0.005	0.005	0.052	0.065	0.065
norm. dec. δ	0	2.54	2.54	2.54	26.27	33.06	33.06
norm. exc. succ. σ	0	2.54	2.54	2.54	26.27	33.06	33.06
SPI Ψ	0	2.54	2.54	2.54	26.27	33.06	33.06
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 28: Two Points, Corr=$\frac{11}{5} - \frac{2}{5}\sqrt{19} \approx 0.456$, $\mathbf{w} = (2, 2, 21, 28, 31, 31)$, $\gamma = 59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	87.93	87.24	87.24	87.24	78.33	75.27	75.27
succ. σ'	87.93	88.08	88.08	88.08	90.09	90.79	90.79
avg. dist. $\overline{d}$	0.121	0.119	0.119	0.119	0.099	0.092	0.092
dec. δ'	0	0.85	0.85	0.85	11.76	15.52	15.52
exc. succ. σ^E	0	0.156	0.156	0.156	2.167	2.859	2.859
red. avg. dist. Ψ'	0	0.002	0.002	0.002	0.022	0.029	0.029
norm. dec. δ	0	1.87	1.87	1.87	25.94	34.22	34.22
norm. exc. succ. σ	0	1.87	1.87	1.87	25.94	34.22	34.22
SPI Ψ	0	1.87	1.87	1.87	25.94	34.22	34.22
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 29: Two Points, Corr=$\frac{1}{3}+\frac{2}{15}\sqrt{5}\approx 0.631$, $\mathbf{w}=(2,2,21,28,31,31)$, $\gamma=59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	92.18	91.82	91.82	91.82	85.46	83.01	83.01
succ. σ'	92.18	92.23	92.23	92.23	93.18	93.54	93.54
avg. dist. $\overline{d}$	0.078	0.078	0.078	0.078	0.068	0.065	0.065
dec. δ'	0	0.42	0.42	0.42	7.71	10.53	10.53
exc. succ. σ^E	0	0.054	0.054	0.054	0.997	1.360	1.360
red. avg. dist. Ψ'	0	0.001	0.001	0.001	0.010	0.014	0.014
norm. dec. δ	0	1.39	1.39	1.39	25.70	35.07	35.07
norm. exc. succ. σ	0	1.39	1.39	1.39	25.70	35.07	35.07
SPI Ψ	0	1.39	1.39	1.39	25.70	35.07	35.07
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 30: Two Points, Corr=$\frac{3}{7}+\frac{2}{35}\sqrt{30}\approx 0.742$, $\mathbf{w}=(2,2,21,28,31,31)$, $\gamma=59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	99.46	99.45	99.45	99.45	98.92	98.66	98.66
succ. σ'	99.46	99.46	99.46	99.46	99.46	99.46	99.46
avg. dist. $\overline{d}$	0.005	0.005	0.005	0.005	0.005	0.005	0.005
dec. δ'	0	0.00	0.00	0.00	0.54	0.81	0.81
exc. succ. σ^E	0	0.000	0.000	0.000	0.006	0.009	0.009
red. avg. dist. Ψ'	0	0.000	0.000	0.000	0.000	0.000	0.000
norm. dec. δ	0	0.13	0.13	0.13	25.07	37.27	37.27
norm. exc. succ. σ	0	0.13	0.13	0.13	25.07	37.27	37.27
SPI Ψ	0	0.13	0.13	0.13	25.07	37.27	37.27
Banzhaf β	0	3.85	3.85	3.85	26.92	30.77	30.77
Shapley-Shubik ϕ	0	3.33	3.33	3.33	26.67	31.67	31.67
ECPI ζ	0	0.00	0.00	0.00	25.00	37.50	37.50

Table 31: Two Points, Corr=$\frac{10}{29}+\frac{1}{145}\sqrt{8445}\approx 0.979$, $\mathbf{w}=(2,2,21,28,31,31)$, $\gamma=59.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	50.00	48.44	48.44	48.44	48.44	48.44	1.56
succ. σ'	50.00	51.56	51.56	51.56	51.56	51.56	98.44
avg. dist. $\overline{d}$	0.500	0.484	0.484	0.484	0.484	0.484	0.016
dec. δ'	0	3.12	3.12	3.12	3.12	3.12	96.88
exc. succ. σ^E	0	1.56	1.56	1.56	1.56	1.56	48.44
red. avg. dist. Ψ'	0	0.016	0.016	0.016	0.016	0.016	0.484
norm. dec. δ	0	2.78	2.78	2.78	2.78	2.78	86.11
norm. exc. succ. σ	0	2.78	2.78	2.78	2.78	2.78	86.11
SPI Ψ	0	2.78	2.78	2.78	2.78	2.78	86.11
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 32: Two Points, Corr=0, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	51.98	50.25	50.25	50.25	50.25	50.25	2.42
succ. σ'	51.98	53.59	53.59	53.59	53.59	53.59	98.17
avg. dist. $\overline{d}$	0.480	0.464	0.464	0.464	0.464	0.464	0.018
dec. δ'	0	3.33	3.33	3.33	3.33	3.33	95.76
exc. succ. σ^E	0	1.61	1.61	1.61	1.61	1.61	46.19
red. avg. dist. Ψ'	0	0.016	0.016	0.016	0.016	0.016	0.462
norm. dec. δ	0	2.97	2.97	2.97	2.97	2.97	85.17
norm. exc. succ. σ	0	2.97	2.97	2.97	2.97	2.97	85.17
SPI Ψ	0	2.97	2.97	2.97	2.97	2.97	85.17
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 33: Two Points, Corr=$\frac{1}{3} - \frac{2}{15}\sqrt{5} \approx 0.035$, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	56.49	54.42	54.42	54.42	54.42	54.42	4.59
succ. σ'	56.49	58.14	58.14	58.14	58.14	58.14	97.64
avg. dist. $\overline{d}$	0.435	0.419	0.419	0.419	0.419	0.419	0.024
dec. δ'	0	3.72	3.72	3.72	3.72	3.72	93.05
exc. succ. σ^E	0	1.65	1.65	1.65	1.65	1.65	41.15
red. avg. dist. Ψ'	0	0.016	0.016	0.016	0.016	0.016	0.411
norm. dec. δ	0	3.33	3.33	3.33	3.33	3.33	83.33
norm. exc. succ. σ	0	3.33	3.33	3.33	3.33	3.33	83.33
SPI Ψ	0	3.33	3.33	3.33	3.33	3.33	83.33
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 34: Two Points, Corr=$\frac{3}{7} - \frac{2}{35}\sqrt{30} \approx 0.116$, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	68.52	65.74	65.74	65.74	65.74	65.74	12.04
succ. σ'	68.52	69.91	69.91	69.91	69.91	69.91	96.76
avg. dist. $\overline{d}$	0.315	0.301	0.301	0.301	0.301	0.301	0.032
dec. δ'	0	4.17	4.17	4.17	4.17	4.17	84.72
exc. succ. σ^E	0	1.39	1.39	1.39	1.39	1.39	28.24
red. avg. dist. Ψ'	0	0.014	0.014	0.014	0.014	0.014	0.282
norm. dec. δ	0	3.95	3.95	3.95	3.95	3.95	80.26
norm. exc. succ. σ	0	3.95	3.95	3.95	3.95	3.95	80.26
SPI Ψ	0	3.95	3.95	3.95	3.95	3.95	80.26
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 35: Two Points, Corr=$\frac{1}{3}$, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	70.30	67.46	67.46	67.46	67.46	67.46	13.37
succ. σ'	70.30	71.61	71.61	71.61	71.61	71.61	96.70
avg. dist. $\overline{d}$	0.297	0.284	0.284	0.284	0.284	0.284	0.033
dec. δ'	0	4.16	4.16	4.16	4.16	4.16	83.33
exc. succ. σ^E	0	1.32	1.32	1.32	1.32	1.32	26.41
red. avg. dist. Ψ'	0	0.013	0.013	0.013	0.013	0.013	0.264
norm. dec. δ	0	3.99	3.99	3.99	3.99	3.99	80.04
norm. exc. succ. σ	0	3.99	3.99	3.99	3.99	3.99	80.04
SPI Ψ	0	3.99	3.99	3.99	3.99	3.99	80.04
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 36: Two Points, Corr=$\frac{2}{15}\sqrt{105} - 1 \approx 0.366$, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	75.08	72.15	72.15	72.15	72.15	72.15	17.29
succ. σ'	75.08	76.17	76.17	76.17	76.17	76.17	96.65
avg. dist. $\overline{d}$	0.249	0.238	0.238	0.238	0.238	0.238	0.034
dec. δ'	0	4.02	4.02	4.02	4.02	4.02	79.36
exc. succ. σ^E	0	1.09	1.09	1.09	1.09	1.09	21.57
red. avg. dist. Ψ'	0	0.011	0.011	0.011	0.011	0.011	0.216
norm. dec. δ	0	4.05	4.05	4.05	4.05	4.05	79.77
norm. exc. succ. σ	0	4.05	4.05	4.05	4.05	4.05	79.77
SPI Ψ	0	4.05	4.05	4.05	4.05	4.05	79.77
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 37: Two Points, Corr=$\frac{11}{5} - \frac{2}{5}\sqrt{19} \approx 0.456$, $\mathbf{w} = (1, 1, 2, 3, 3, 10)$, $\gamma = 10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	83.95	81.23	81.23	81.23	81.23	81.23	26.10
succ. σ'	83.95	84.56	84.56	84.56	84.56	84.56	97.01
avg. dist. $\overline{d}$	0.161	0.154	0.154	0.154	0.154	0.154	0.030
dec. δ'	0	3.33	3.33	3.33	3.33	3.33	70.91
exc. succ. σ^E	0	0.614	0.614	0.614	0.614	0.614	13.066
red. avg. dist. Ψ'	0	0.006	0.006	0.006	0.006	0.006	0.131
norm. dec. δ	0	3.81	3.81	3.81	3.81	3.81	80.97
norm. exc. succ. σ	0	3.81	3.81	3.81	3.81	3.81	80.97
SPI Ψ	0	3.81	3.81	3.81	3.81	3.81	80.97
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 38: Two Points, Corr=$\frac{1}{3}+\frac{2}{15}\sqrt{5}\approx 0.631$, $\mathbf{w}=(1,1,2,3,3,10)$, $\gamma=10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	89.16	86.90	86.90	86.90	86.90	86.90	32.47
succ. σ'	89.16	89.50	89.50	89.50	89.50	89.50	97.58
avg. dist. $\overline{d}$	0.108	0.105	0.105	0.105	0.105	0.105	0.024
dec. δ'	0	2.60	2.60	2.60	2.60	2.60	65.11
exc. succ. σ^E	0	0.337	0.337	0.337	0.337	0.337	8.414
red. avg. dist. Ψ'	0	0.003	0.003	0.003	0.003	0.003	0.084
norm. dec. δ	0	3.33	3.33	3.33	3.33	3.33	83.33
norm. exc. succ. σ	0	3.33	3.33	3.33	3.33	3.33	83.33
SPI Ψ	0	3.33	3.33	3.33	3.33	3.33	83.33
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 39: Two Points, Corr=$\frac{3}{7}+\frac{2}{35}\sqrt{30}\approx 0.742$, $\mathbf{w}=(1,1,2,3,3,10)$, $\gamma=10.5$

	Pl. 0	Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5	Pl. 6
luck λ	99.19	98.93	98.93	98.93	98.93	98.93	48.41
succ. σ'	99.19	99.19	99.19	99.19	99.19	99.19	99.74
avg. dist. $\overline{d}$	0.008	0.008	0.008	0.008	0.008	0.008	0.003
dec. δ'	0	0.26	0.26	0.26	0.26	0.26	51.33
exc. succ. σ^E	0	0.003	0.003	0.003	0.003	0.003	0.549
red. avg. dist. Ψ'	0	0.000	0.000	0.000	0.000	0.000	0.005
norm. dec. δ	0	0.50	0.50	0.50	0.50	0.50	97.50
norm. exc. succ. σ	0	0.50	0.50	0.50	0.50	0.50	97.50
SPI Ψ	0	0.50	0.50	0.50	0.50	0.50	97.50
Banzhaf β	0	2.78	2.78	2.78	2.78	2.78	86.11
Shapley-Shubik ϕ	0	3.33	3.33	3.33	3.33	3.33	83.33
ECPI ζ	0	0.00	0.00	0.00	0.00	0.00	100.00

Table 40: Two Points, Corr=$\frac{10}{29}+\frac{1}{145}\sqrt{8445}\approx 0.979$, $\mathbf{w}=(1,1,2,3,3,10)$, $\gamma=10.5$

B Proofs

B.1 Theorem 1

Proof. First, note that $A_0(q,p,X_0),\dots,A_n(q,p,X_n)$ are, conditional on $Q=q, P=p$, independent Bernoulli variables taking the value 1 with probability $\pi(q,p)$ each, whereas, unconditionally, they are possibly dependent Bernoulli variables with mean $\frac{1}{2}$ and variance $\frac{1}{4}$ each. We therefore have for $i \neq j$:

$$\begin{aligned}
&\operatorname{Cov}(A_i(Q,P,X_i), A_j(Q,P,X_j)) \\
=&E\big(A_i(Q,P,X_i)A_j(Q,P,X_j)\big) - E\big(A_i(Q,P,X_i)\big)E\big(A_j(Q,P,X_j)\big) \\
=&\operatorname{Prob}\big(A_i(Q,P,X_i)=1 \wedge A_j(Q,P,X_j)=1\big) - \frac{1}{2}\cdot\frac{1}{2} \\
=&\int_{X^2_{\neq}} \operatorname{Prob}\big(A_i(Q,P,X_i)=1 \wedge A_j(Q,P,X_j)=1 | (Q,P)=(q,p)\big)\nu(d(q,p)) - \frac{1}{4} \\
=&\int_{X^2_{\neq}} \pi(q,p)^2\nu(d(q,p)) - \frac{1}{4} \\
=&E(\pi(Q,P)^2) - E(\pi(Q,P))^2 \\
=&\operatorname{Var}(\pi(Q,P)) = \operatorname{Var}(\Pi).
\end{aligned}$$

The formula for the correlation is an immediate consequence of $\operatorname{Var}(A(Q,P,X_i)) = \frac{1}{4}$ for all i. □

B.2 Corollary 4

Proof. Notice first that

$$a^j b^k + a^k b^j = \min(a,b)^j \max(a,b)^k + \min(a,b)^k \max(a,b)^j$$

for all numbers a, b. Furthermore,

$$\max(\pi(Q,P), 1-\pi(Q,P)) = \frac{1}{2}(1+|2\pi(Q,P)-1|) = \frac{1}{2}\left(1+\sqrt{\xi(Q,P)}\right),$$

$$\min(\pi(Q,P), 1-\pi(Q,P)) = \frac{1}{2}(1-|2\pi(Q,P)-1|) = \frac{1}{2}\left(1-\sqrt{\xi(Q,P)}\right).$$

Together, this implies

$$\pi(Q,P)^j(1-\pi(Q,P))^k + \pi(Q,P)^k(1-\pi(Q,P))^j =$$
$$\left(\frac{1}{2}\right)^{j+k}\left(\left(1-\sqrt{\xi(Q,P)}\right)^j\left(1+\sqrt{\xi(Q,P)}\right)^k + \left(1-\sqrt{\xi(Q,P)}\right)^k\left(1+\sqrt{\xi(Q,P)}\right)^j\right)$$

Additionally, we have $\pi(q,p) = 1-\pi(p,q)$ which implies $z(q,p) = z(p,q)$. As $(Q,P) \overset{d}{=} (P,Q)$, we also have

$$w_{j,k,l} = E\left(\pi(Q,P))^j(1-\pi(Q,P))^k z(Q,P)^l\right) = E\left((1-\pi(Q,P))^j\pi(Q,P)^k z(Q,P)^l\right).$$

Combining all these, we arrive at the formula given in the Corollary. □

B.3 Proposition 3

Proof. 1. This is a direct consequence of 1. of Corollary 4.

2. By using the definition and factoring out, we get for part (a):

$$\begin{aligned}
&w_{m+1,\tilde{n}-m,l} + w_{m,\tilde{n}+1-m,l} \\
=&E\left(\pi(Q,P)^{m+1}\left(1-\pi(Q,P)\right)^{\tilde{n}-m} z(Q,P)^{l}\right) \\
&+ E\left(\pi(Q,P)^{m}\left(1-\pi(Q,P)\right)^{\tilde{n}+1-m} z(Q,P)^{l}\right) \\
=&E\left(\pi(Q,P)^{m}\left(1-\pi(Q,P)\right)^{\tilde{n}-m}(\pi(Q,P)+1-\pi(Q,P))z(Q,P)^{l}\right) \\
=&w_{m,\tilde{n}-m,l}
\end{aligned}$$

With regard to (b), notice that, with the abbreviation $\xi := \xi(Q,P)$,

$$\begin{aligned}
&\left(1-\sqrt{\xi}\right)^{m}\left(1+\sqrt{\xi}\right)^{\tilde{n}-m} + \left(1-\sqrt{\xi}\right)^{\tilde{n}-m}\left(1+\sqrt{\xi}\right)^{m} \\
-&\left(\left(1-\sqrt{\xi}\right)^{m+1}\left(1+\sqrt{\xi}\right)^{\tilde{n}-m-1} + \left(1-\sqrt{\xi}\right)^{\tilde{n}-m-1}\left(1+\sqrt{\xi}\right)^{m+1}\right) \\
=&2\sqrt{\xi}\left(\left(1-\sqrt{\xi}\right)^{m}\left(1+\sqrt{\xi}\right)^{\tilde{n}-m-1} - \left(1-\sqrt{\xi}\right)^{\tilde{n}-m-1}\left(1+\sqrt{\xi}\right)^{m}\right)
\end{aligned}$$

which is, uniformly in $\xi \in [0,1]$, non-negative (non-positive) if and only if $m \leq \frac{\tilde{n}-1}{2}$ ($m \geq \frac{\tilde{n}-1}{2}$). The assertion then follows by combining this result with the formula for w given in 1. of Corollary 4.

3. These, as well as 4., are easy but tedious consequences of the definitions and the fact that

$$|N \setminus (M \cup \{i\})| = n - |M|.$$

□